Lie algebras

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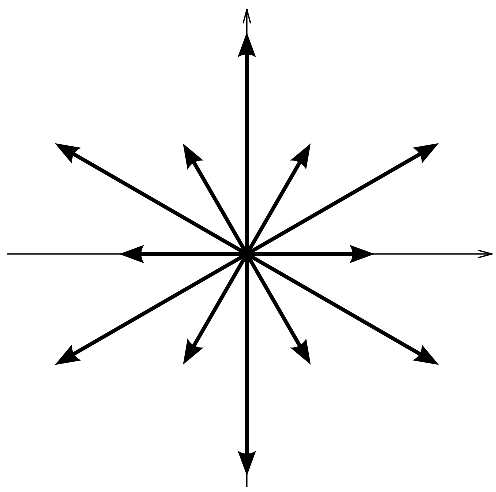
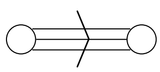




Contents

Main references used:

- Section 1: Baker, Matrix Groups (Springer 2002)
- Sections 2–6: Humphreys, Introduction to Lie algebras and representation theory (Springer 1972)
- Sections 5, 6: Kirillov, An introduction to Lie groups and Lie algebras (CUP 2008)

1 Matrix Lie groups and Lie algebras

1.1 Preliminary remarks (for Info)

Definition 1.1.1. A *Lie group* is a topological space G equipped with the structure of a group and of a smooth manifold, such that the maps $g \mapsto g^{-1}$ and $(g, h) \mapsto g \cdot h$ are smooth.

Definition 1.1.2. Let k be a field. A *Lie algebra (over k)* is a k-vector space $\mathfrak{g}$ together with a bilinear map

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} ,$$

the *Lie bracket*, such that

1. *(skew-symmetry)* $[x, x] = 0$ for all $x \in \mathfrak{g}$,
2. *(Jacobi identity)* $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

Remark 1.1.3.
1. If the characteristic of k is not 2, skew-symmetry is equivalent to $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$. (Why?) ?
2. A Lie group G gives rise to a Lie algebra on the tangent space T_eG at the neutral element $e \in G$.
3. Symmetries in physics:

$$\begin{array}{ccc} \text{state at time } 0 & \xrightarrow{\text{evolve}} & \text{state at time } t \\ \downarrow f & & \downarrow f \\ \text{state}' \text{ at time } 0 & \xrightarrow{\text{evolve}} & \text{state}' \text{ at time } t \end{array}$$

   Symmetries are often Lie groups.

1.2 Rotations and translations

■ Consider $\mathbb{R}^n$ with inner product $g(u,v) = \sum_{i=1}^n u_i v_i$. A linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is an *orthogonal transformation* if $g(Tu, Tv) = g(u,v)$ for all $u, v \in \mathbb{R}^n$. In the standard basis $\{e_i\}_{i=1,\dots,n}$ of $\mathbb{R}^n$ and the matrix representation $T_{ij} = (Te_i)_j$, this is equivalent to $T^t T = \mathbb{1}$ (Why?). T is a *rotation* if in addition $\det(T) = 1$. ?

■ Examples of groups:

- *Orthogonal group:* $O(n) = O(n,\mathbb{R}) = \{M \in \mathrm{Mat}(n,\mathbb{R}) \,|\, M^t M = \mathbb{1}\}$.
- *Special orthogonal group:* $SO(n) = \{M \in O(n) \,|\, \det(M) = 1\}$.
- *Euclidean group:*

$$E(n) = \{\text{maps } f : \mathbb{R}^n \to \mathbb{R}^n \,|\, \forall x, y \in \mathbb{R}^n \colon |x-y| = |f(x) - f(y)|\} \ .$$

$O(n)$ and $SO(n)$ are groups (Why?). The maps $f \in E(n)$ are precisely those of the form $f(x) = Tx + u$ with $T \in O(n)$ and $u \in \mathbb{R}^n$ (Exercise). ? E

Definition 1.2.1. Let H, N be groups, and $\varphi : H \to \mathrm{Aut}(N)$, $h \mapsto \varphi_h$ a group homomorphism from H into the group automorphisms of N. The *semidirect product* $H \ltimes_\varphi N = H \ltimes N$ is the set $H \times N$ with product and inverse (Sheet 1)

$$(h,n) \cdot (h',n') := (h \cdot h', n \cdot \varphi_h(n')) \quad , \quad (h,n)^{-1} := (h^{-1}, \varphi_{h^{-1}}(n^{-1})) \ .$$

Remark 1.2.2. 1. $H \rtimes N$ is a group. (Why?) ?

2. If $\varphi_h = \mathrm{id}_N$ for all $h \in H$, then $H \rtimes N$ is the *direct product $H \times N$ of groups*.

3. $1 \xrightarrow{f_1} N \xrightarrow{f_2} H \ltimes N \xrightarrow{f_3} H \xrightarrow{f_4} 1$, with $f_2(n) = (1,n)$ and $f_3(h,n) = h$, is an *exact sequence of groups*, i.e. $\mathrm{im}\, f_i = \ker f_{i+1}$, $i = 1,2,3$.

Example 1.2.3. For $f, f' \in E(n)$, $f(x) = Tx + u$, etc, have $f(f'(x)) = f(T'x + u') = T(T'x + u') + u = TT'x + Tu' + u$. In fact

$$E(n) \to O(n) \ltimes_\varphi \mathbb{R}^n \quad , \quad f(x) = Tx + u \mapsto (T,u)$$

is a group isomorphism for $\varphi : O(n) \to \mathrm{Aut}(\mathbb{R}^n)$, $\varphi_T(x) = Tx$.

- n-dimensional *Minkowski space* is $\mathbb{R}^n$ with the non-degenerate bilinear form

$$\eta(u,v) = u_0 v_0 - u_1 v_1 - \cdots - u_{n-1} v_{n-1}$$

where the entries of $u \in \mathbb{R}^n$ are numbered from 0 to $n-1$,

$$u = (\underbrace{u_0}_{\text{time}}, \underbrace{u_1, \dots, u_{n-1}}_{\text{space}}) \ .$$

- Further examples of groups:

  - *Lorentz group:*

$$O(1,n-1) = \{M \in \mathrm{Mat}(n,\mathbb{R}) \,|\, \forall x,y \in \mathbb{R}^n \colon \eta(Mx,My) = \eta(x,y)\} \ .$$

  - *Poincaré group:*

$$P(1,n-1) = \{f : \mathbb{R}^n \to \mathbb{R}^n \,|\, f(x) = \Lambda x + u \text{ for some } \Lambda \in O(1,n-1), u \in \mathbb{R}^n\} \ .$$

We have $O(1,n-1) = \{M \in \mathrm{Mat}(n,\mathbb{R}) \,|\, M^t J M = J\}$, where J is the diagonal $n \times n$ matrix with entries $1, -1, \cdots -1$ on the diagonal (Exercise). E
Analogous to $E(n)$ one checks $P(1,n-1) \cong O(1,n-1) \rtimes \mathbb{R}^n$ (Details?). (Sheet 1)

?

1.3 Matrix Lie groups

- The *general linear group* of a vector space V is

$$GL(V) = \{\text{invertible linear maps } V \to V\} \ .$$

Write $GL(n,\mathbb{R})$ for $GL(\mathbb{R}^n)$.

Definition 1.3.1. A *matrix Lie group* is a closed subgroup of $GL(N,\mathbb{R})$ or $GL(n,\mathbb{C})$ ($n \geq 1$).

Here, "closed" is meant in the topological sense (the complement is an open set).

Information: Every matrix Lie group is a Lie group in the sense of Definition 1.1.1. However, not every Lie group is isomorphic to a matrix Lie group.

Example 1.3.2. 1. $GL(n,\mathbb{R})$, $O(n)$, $SO(n)$, $O(1,n-1)$ are matrix Lie groups. For example, the map $f : \mathrm{Mat}(n,\mathbb{R}) \to \mathrm{Mat}(n,\mathbb{R})$, $M \mapsto M^t M$ is continuous, and so $O(n) = f^{-1}(\{\mathbb{1}\})$ is the preimage of a closed set, hence closed.

2. $E(n)$ and $P(1, n-1)$ are isomorphic to matrix Lie groups. This can be seen via the group homomorphism (Exercise) Ⓔ (Ex. 1)

$$GL(n,\mathbb{R}) \rtimes \mathbb{R}^n \to \mathrm{Mat}(n+1,\mathbb{R}) \quad , \quad (T,u) \mapsto \begin{pmatrix} T & u \\ 0 & 1 \end{pmatrix} \ .$$

3. *Special linear group:* For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, set (17.10.)

$$SL(n,\mathbb{K}) = \{M \in \mathrm{Mat}(n,\mathbb{K}) \,|\, \det M = 1\} \ .$$

4. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and set

$$J_{\mathrm{sp}} = \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix} \in \mathrm{Mat}(2n,\mathbb{R}) \ .$$

The *symplectic group* is

$$SP(2n,\mathbb{K}) = \{A \in \mathrm{Mat}(2n,\mathbb{K}) \,|\, A^t J_{\mathrm{sp}} A = J_{\mathrm{sp}}\} \ .$$

(For info: The defining condition implies at once that $\det A = \pm 1$. It is less obvious that $A \in SP(2n,\mathbb{K})$ already implies $\det A = 1$. One way to see this is to 1) note that A preserves a symplectic bilinear form on $\mathbb{K}^n$, 2) recall that the determinant of a linear map $V \to V$ is given by its action on alternating top forms, and 3) build a top form out of a symplectic basis.

5. On $\mathbb{C}^n$ define the inner product $(u,v) = \sum_{j=1}^n u_j^* v_j$. The *unitary group* is

$$U(n) = \{A \in \mathrm{Mat}(n,\mathbb{C}) \,|\, \forall u,v \in \mathbb{C}^n \colon (Au,Av) = (u,v)\} \ ,$$

and the *special unitary group* is

$$SU(n) = \{A \in U(n) \,|\, \det A = 1\} \ .$$

$SL(n,\mathbb{K})$, $SP(2n,\mathbb{K})$, $U(n)$, $SU(n)$ are matrix Lie groups. (Exercise.) Ⓔ (Sheet 1)

1.4 Lie algebra, general definition

k : commutative ring (e.g. $k = \mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{C}[X]$, ...)

Definition 1.4.1. A k-*algebra* is a pair (A,b) where A is a k-module and $b : A \times A \to A$ is a k-bilinear map.

■ A k-algebra A is *associative* if $b(b(x,y),z) = b(x,b(y,z))$ for all $x,y,z \in A$.

Definition 1.4.2. A *Lie algebra over k* is a k-algebra $L = (L, [-,-])$ such that

1. *(skew-symmetry)* $[x, x] = 0$ for all $x \in L$,
2. *(Jacobi identity)* $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

Example 1.4.3.

1. For any k-module L one can define $[a, b] = 0$ for all $a, b \in L$. A Lie algebra with zero Lie bracket is called *abelian*.
2. Let $(A, \cdot)$ be an associative algebra. The *commutator* is the map $[-,-] : A \times A \to A$, $[a, b] = a \cdot b - b \cdot a$. Then $(A, [-,-])$ is a Lie algebra. (Why?) ?
3. Let $A = (A, b)$ be a k-algebra. A *derivation on A* is a k-linear map $D : A \to A$ such that $D(b(x, y)) = b(Dx, y) + b(x, Dy)$. Write

$$\mathrm{Der}(A) := \{D | D \text{ is a derivation on } A\} .$$

   $\mathrm{Der}(A)$ is a Lie algebra with Lie bracket $[D, E] = D \circ E - E \circ D$. (Why ? is $[D, E]$ again a derivation? Give an example where $D \circ E$ is not a derivation.)

Definition 1.4.4. A *(Lie) algebra homomorphism from L to K* is a k-linear map $\varphi : L \to K$ such that

$$\varphi([x, y]) = [\varphi(x), \varphi(y)] \quad \text{for all} \quad x, y \in L .$$

Definition 1.4.5. A *Lie subalgebra of L* is a sub-k-module $K \subset L$ (a k-linear subspace), such that for all $x, y \in K$ also $[x, y] \in K$.

■ From here on we will only consider Lie algebras over a field k (rather than a general commutative ring).

1.5 Lie algebras from matrix Lie groups

■ Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The *exponential* of a matrix $X \in \mathrm{Mat}(n, \mathbb{K})$ is

$$\exp(X) := \sum_{k=0}^{\infty} \frac{1}{k!} X^k = \mathbb{1} + X + \tfrac{1}{2} X^2 + \tfrac{1}{6} X^3 + \cdots$$

By convention, $X^0 := \mathbb{1}$, even for $X = 0$.

Remark 1.5.1.

1. The matrix exponential converges absolutely for all X. (Why? Hint: Use the operator norm on $\mathrm{Mat}(n, \mathbb{K})$, or any other submultiplicative norm.) ?

2. Let $X, Y \in \mathrm{Mat}(n, \mathbb{K})$. If $XY = YX$, then $\exp(X)\exp(Y) = \exp(X + Y) = \exp(Y)\exp(X)$. It follows that $\exp(X) \in GL(n, \mathbb{K})$ for all $X \in \mathrm{Mat}(n, \mathbb{K})$.

3. $\det(\exp(X)) = \exp(\mathrm{tr}(X))$ (Exercise.) E (Ex. 2)

Example 1.5.2. 1. For $\lambda_1, \dots, \lambda_n \in \mathbb{C}$,

$$\exp\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix}$$

2. For $\lambda \in \mathbb{C}$,

$$\exp\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = e^{\lambda}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad , \quad \exp\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix} = \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix}$$

Proposition 1.5.3. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. There exist open neighbourhoods U of 0 and V of $\mathbb{1}$ in $\mathrm{Mat}(n, \mathbb{K})$ such that $\exp : U \to V$ is a diffeomorphism.

Sketch of proof. Consider $\mathbb{K} = \mathbb{R}$, the argument for $\mathbb{C}$ is similar. First one checks that the total derivative of exp exists everywhere on $\mathrm{Mat}(n, \mathbb{R})$. Then one computes the total derivative at 0 explicitly to see that it is invertible. The inverse function theorem gives the result. (Alternatively one can define the matrix log and show that it is inverse to exp in a suitable neighbourhood.) □

Definition 1.5.4. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and $G \subset GL(n, \mathbb{K})$ a matrix Lie group. The *Lie algebra of G* is

$$\mathfrak{g} = \{A \in \mathrm{Mat}(n, \mathbb{K}) \,|\, \exp(sA) \in G \text{ for all } s \in \mathbb{R}\} \ .$$

Theorem 1.5.5 (for information)**.** Let G, $\mathfrak{g}$ be as in the above definition.

1. $\mathfrak{g}$ is a Lie algebra with Lie bracket given by the commutator in $\mathrm{Mat}(n, \mathbb{K})$.

2. There are open neighbourhoods U of $0 \in \mathfrak{g}$ and V of $\mathbb{1} \in G$ such that $\exp : U \to V$ is a diffeomorphism.

For a proof see Thm. 7.32 in Baker, "Matrix Groups", Springer.

Remark 1.5.6 (for information)**.** 1. Slogan: The Lie algebra $\mathfrak{g}$ knows all about the Lie group G near the unit $e \in G$. Indeed, the product on G can be recovered from the Lie bracket on $\mathfrak{g}$ via the *Baker-Campbell-Hausdorff formula*: $\exp(X)\exp(Y) = \exp(X * Y)$, where

$$X * Y = X + Y + \tfrac{1}{2}[X, Y] + \tfrac{1}{12}[X, [X, Y]] + \tfrac{1}{12}[Y, [Y, X]] + \cdots$$

2. For a general Lie group G (in the sense of Definition 1.1.1), the Lie algebra $\mathfrak{g}$ of G is $\mathfrak{g} = T_e G$ as a $\mathbb{R}$-vector space and the Lie bracket is defined in terms of vector fields on G as follows: Let $L_a : G \to G$, $g \mapsto ag$ (left translation). A vector field X on G is called *left invariant* if $(DL_a)(b)X(b) = X(b)$ for all $b \in G$, i.e. $(L_a)_* X = X$. Let $\mathcal{L}$ be the $\mathbb{R}$-vector space of all left-invariant vector fields on G. One checks that for $X, Y \in \mathcal{L}$ also $[X, Y] \in \mathcal{L}$. Finally, $\mathcal{L} \to T_e G$, $X \mapsto X(e)$ is an $\mathbb{R}$-linear isomorphism.

Example 1.5.7. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$.

Mat. Lie grp.	Lie algebra
$GL(n,\mathbb{K})$	$gl(n,\mathbb{K}) = \mathrm{Mat}(n,\mathbb{K})$
$SL(n,\mathbb{K})$	$sl(n,\mathbb{K}) = \{A \in \mathrm{Mat}(n,\mathbb{K}) \mid \mathrm{tr}\, A = 0\}$
$SP(2n,\mathbb{K})$	$sp(2n,\mathbb{K}) = \{A \in \mathrm{Mat}(n,\mathbb{K}) \mid A^t J_{\mathrm{sp}} + J_{\mathrm{sp}} A = 0\}$
$O(n)$	$o(n) = \{A \in \mathrm{Mat}(n,\mathbb{R}) \mid A^t + A = 0\}$
$O(1,n-1)$	$o(1,n-1) = \{A \in \mathrm{Mat}(n,\mathbb{R}) \mid A^t J + JA = 0\}$
$SO(n)$	$so(n) = o(n)$
$U(n)$	$u(n) = \{A \in \mathrm{Mat}(n,\mathbb{C}) \mid A^\dagger + A = 0\}$
$SU(n)$	$su(n) = \{A \in u(n) \mid \mathrm{tr}\, A = 0\}$

Some examples of how to verify this are:

- $GL(n,\mathbb{K})$: e^{sA} is invertible for all choices of s, A.
- $SL(n,\mathbb{K})$: $A \in sl(n,\mathbb{K}) \overset{Def}{\Leftrightarrow} e^{sA} \in SL(n,\mathbb{K})$ for all $s \in \mathbb{R} \Leftrightarrow \det e^{sA} = 1$ for all $s \in \mathbb{R}$. Since $\det e^{sA} = e^{s\,\mathrm{tr}\, A}$, this happens iff $\mathrm{tr}\, A = 0$.
- $O(n)$: If $(e^{sA})^t e^{sA} = \mathbb{1}$ for all $s \in \mathbb{R}$, differentiating with respect to s gives $A^t + A = 0$. Conversely, if $A^t + A = 0$, then in particular $AA^t = A^t A$ and so $(e^{sA})^t e^{sA} = e^{sA^t + sA} = \mathbb{1}$.

Check explicitly that these are Lie algebras: For example in the case of $o(n)$, let $A, B \in o(n)$ and set $C = [A, B] = AB - BA$. We need to check that also $C \in o(n)$. Indeed: (20.10.)

$$C^t = (AB - BA)^t = B^t A^t - A^t B^t = (-B)(-A) - (-A)(-B) = [B, A] = -C\ .$$

1.6 Example: $so(3)$ and $su(2)$

■ Notation: $E_{ij} = E(n)_{ij}$ is the $n \times n$ matrix with a 1 in row i and column j and zeros else: $(E_{ij})_{ab} = \delta_{i,a}\delta_{j,b}$. E.g.

$$E(3)_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ .$$

We have $E_{ab}E_{cd} = \delta_{b,c}E_{ad}$.

The Lie algebra $so(3)$

A basis of $so(3)$ is given by

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad , \quad J_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad , \quad J_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ .$$

J_i is the generator of rotations around the i-axis. E.g. $\exp(-\theta J_1)$ is a rotation in the 2-3-plane by angle θ, cf. Example 1.5.2.

■ Notation: ε_{abc} denotes the totally antisymmetric tensor for $a, b, c \in \{1, 2, 3\}$, normalised such that $\varepsilon_{123} = 1$. Then $\varepsilon_{321} = -1$, $\varepsilon_{323} = 0$, etc. We have (Why?) ?

$$\sum_{x=1}^{3} \varepsilon_{abx}\, \varepsilon_{cdx} = \delta_{a,c}\delta_{b,d} - \delta_{a,d}\delta_{b,c} \ .$$

Using ε_{abc} the above basis of $so(3)$ becomes $J_a = \sum_{b,c=1}^{3} \varepsilon_{abc} E_{bc}$ (Why?) ?
and the Lie bracket of basis elements is (Exercise): E (Ex. 3)

$$[J_a, J_b] = -\sum_{c=1}^{3} \varepsilon_{abc}\, J_c \ .$$

The Lie algebra $su(2)$

The *Pauli matrices* are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

Note that $\sigma_a^\dagger = \sigma_a$ and $\mathrm{tr}\, \sigma_a = 0$. $su(2)$ consists of all $A \in \mathrm{Mat}(2, \mathbb{C})$ such that $A^\dagger + A = 0$ and $\mathrm{tr}\, A = 0$. A basis (over $\mathbb{R}$) of $su(2)$ is given by $\{i\sigma_1, i\sigma_2, i\sigma_3\}$. The Lie bracket is (Exercise) E (Ex. 3)

$$[i\sigma_a, i\sigma_b] = -2\sum_{c=1}^{3} \varepsilon_{abc}\, i\sigma_c \ .$$

Remark 1.6.1. 1. The Lie algebras $so(3)$ and $su(2)$ are isomorphic as real Lie algebras. (Exercise)

E

(Ex. 3)

2. (for info) The Lie groups $SO(3)$ and $SU(2)$ are not isomorphic as Lie groups (there is no continuous group isomorphism). Indeed, $SU(2)$ is simply connected, $SO(3)$ is not. So there is not even a homeomorphism, irrespective of the group structure.

2 Lie algebras and representations

In this section, k denotes a field.

2.1 Ideals

Definition 2.1.1. Let L be a Lie algebra over k. An *ideal in* L is a sub-vector space $I \subset L$ such that

$$\forall x \in L, a \in I : [x, a] \in I \ .$$

Remark 2.1.2. For subsets $A, B \subset L$ write

$$[A, B] = \mathrm{span}_k\{[a, b] \,|\, a \in A, b \in B\} \ .$$

Then $H \subset L$ is a Lie subalgebra of L iff $[H, H] \subset H$, and an ideal in L iff $[L, H] \subset H$.

Example 2.1.3. 1. $sl(n, k) \subset gl(n, k)$ is an ideal. Indeed, if $x \in gl(n, k)$ and $a \in sl(n, k)$, then $\mathrm{tr}([x, a]) = \mathrm{tr}(xa - ax) = 0$, and so $[x, a] \in sl(n, k)$.

2. $I = \{\lambda \mathbb{1} | \lambda \in k\} \subset gl(n, k)$ is an ideal. Indeed, for all $x \in gl(n, k)$, $[x, \lambda \mathbb{1}] = 0 \in I$.

3. Write $\widetilde{P}(1, n-1)$ for the matrix Lie group given by the image of the Poincaré group $P(1, n-1)$ in $\mathrm{Mat}(n+1, \mathbb{R})$ via Example 1.3.2. The Lie algebra of $\widetilde{P}(1, n-1)$ is (Exercise)

E

(Ex. 5)

$$p(1, n-1) = \left\{ \begin{pmatrix} A & x \\ 0 & 0 \end{pmatrix} \middle| A \in o(1, n-1), x \in \mathbb{R}^n \right\}$$

Then the elements $\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)$ for an ideal of $p(1, n-1)$. (Why?)

?

4. $\left\{\left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) \middle| a \in k\right\} \subset \mathrm{Mat}(2,k)$ is an ideal in $\left\{\left(\begin{smallmatrix} b & a \\ 0 & c \end{smallmatrix}\right) \middle| a,b,c \in k\right\}$, but not in $gl(2,k)$. But it is a Lie subalgebra in both. (Details?) ?

■ Let K, L be Lie algebras over k and $\varphi : K \to L$ a Lie algebra homomorphism. The *kernel of* φ and the image $\mathrm{im}(\varphi)$ are

$$\ker\varphi = \{x \in K \,|\, \varphi(x) = 0\} \subset K \quad , \quad \mathrm{im}\,\varphi = \{\varphi(x) \,|\, x \in K\} \subset L \ .$$

Lemma 2.1.4. For a Lie algebra homomorphism $\varphi : K \to L$, $\ker\varphi$ is an ideal in K.

Proof. Let $a \in \ker\varphi$ and $x \in K$. We need to show that $[x,a] \in \ker\varphi$. This follows from

$$\varphi([x,a]) = [\varphi(x), \varphi(a)] = [\varphi(x), 0] = 0 \ ,$$

where we used that $\varphi(a) = 0$ as $a \in \ker\varphi$. □

(Can you give an example where $\mathrm{im}(\varphi)$ is not an ideal? And show that it is always a Lie subalgebra?) ?

Remark 2.1.5. 1. The *centre of* L is

$$Z(L) = \{z \in L \,|\, [x,z] = 0 \text{ for all } x \in L\} \ .$$

$Z(L)$ is an ideal in L. L is abelian (recall Example 1.4.3) iff $Z(L) = L$.

2. The *derived algebra of* L is $[L,L]$. The derived algebra is an ideal in L.

3. If $I, J \subset L$ are ideals, then $I \cap J$ and $I + J = \{i + j \,|\, i \in I, j \in K\}$ are also ideals in L. (Why?) ?

Proposition 2.1.6. Let L be a Lie algebra over k and $I \subset L$ an ideal.

1. The quotient vector space L/I is a Lie algebra with Lie bracket

$$[x + I, y + I]_{L/I} := [x,y]_L + I \ .$$

2. The canonical projection $\pi : L \to L/I$, $x \mapsto x + I$, is a Lie algebra homomorphism.

Proof. Part 1:

Claim: $[-,-]_{L/I}$ is well-defined.

⌈ We need to check that the definition is independent of the choice of representative. Let $x' = x + i$, $y = y + j$ with $i, j \in I$.

$$[x', y']_L = [x+i, y+j]_L = [x, y]_L + \underbrace{[x, j]_L}_{\in I} + \underbrace{[i, y]_L}_{\in I} + \underbrace{[i, j]_L}_{\in I} \ .$$

Thus $[x + y]_L + I = [x', y']_L + I$, as required. ⌋

Claim: $[-,-]_{L/I}$ satisfies skew-symmetry and the Jacobi identity.

⌈ Skew-symmetry: $[x + I, x + I]_{L/I} = [x, x]_L + I = 0 + I$.
Jacobi identity: analogously. (Details?) ⌋ ?

Part 2: The equality $\pi([x, y]_L) = [\pi(x), \pi(y)]_{L/I}$ is immediate from the definitions. □

Definition 2.1.7. A Lie algebra L is called *simple* if its only ideals are $\{0\}$ and L and if L is not abelian.

(Can you find a non-zero abelian Lie algebra whose only ideals are $\{0\}$ and L?) ?

Lemma 2.1.8. If L is simple, then $Z(L) = \{0\}$.

Proof. Since $Z(L)$ is an ideal, we have $Z(L) = \{0\}$ or $Z(L) = L$. But if $Z(L) = L$, then L is abelian. □

Example: $\boldsymbol{sl(2, k)}$

A basis of $sl(2, k)$ is given by $\{e, f, h\}$ with

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

The Lie brackets are

$$[h, e] = 2e \quad , \quad [h, f] = -2f \quad , \quad [e, f] = h \ .$$

Proposition 2.1.9. Suppose that $\mathrm{char}(k) \neq 2$. Then $sl(2, k)$ is simple.

Proof. Let $I \neq \{0\}$ be an ideal in $sl(2, k)$. We will show that $I = sl(2, k)$. Let $v \in I$ be non-zero. Write $v = ae + bf + ch$ with $a, b, c \in k$. Then

$$[e, v] = bh - 2ce \quad , \qquad [e, [e, v]] = -2be \quad ,$$
$$[f, v] = -ah + 2cf \quad , \qquad [f, [f, v]] = -2af \ .$$

Thus, if $a \neq 0$, then also $f \in I$, and taking repeated brackets with e shows that also $h, e \in I$, and hence $I = sl(2,k)$. Analogously, $b \neq 0$ implies $I = sl(2,k)$.

If $a = b = 0$, then $c \neq 0$ and so $h \in I$. Taking Lie brackets with e and f again shows that also $e, f \in I$, and hence $I = sl(2,k)$. □

(For $\mathrm{char}(k) = 2$, $sl(2,k)$ is not simple. Can you find a non-trivial ideal?) ?

2.2 Representations

In this subsection, L denotes a Lie algebra over k.

■ For a k-vector space V, $gl(V)$ denotes the Lie algebra $(\mathrm{End}(V), [-,-])$. In other words, $gl(V)$ is the vector space of all linear maps $V \to V$ with Lie bracket given by the commutator $[A,B] = AB - BA$.

Definition 2.2.1. 1. A *representation of* L is a pair (V,R) where V is a k-vector space and $R : L \to gl(V)$ is a Lie algebra homomorphism.

2. An *L-module* is a pair (V,ρ) where $\rho : L \times V \to V$ is a bilinear map such that

$$a.(b.v) - b.(a.v) = [a,b]_L.v \qquad \text{for all } a,b \in L,\ v \in V\ ,$$

where $a.v := \rho(a,v)$ and $[-,-]_L$ denotes the Lie bracket of L.

Remark 2.2.2. Let V be a k-vector space. There is a 1-1 correspondence between representations of L on V and L-module structures on V via

$$\rho(a,v) = R(a)v\ .$$

For example, if V is an L-module, then for all $v \in V$,

$$\begin{aligned} R([a,b]_L)v = \rho([a,b]_L, v) = [a,b]_L.v &= a.(b.v) - b.(a.v) \\ &= R(a)R(b)v - R(b)R(a)v = [R(a),R(b)]v\ , \end{aligned}$$

and so R is a Lie algebra homomorphism $L \to gl(V)$. (What is the argument for the converse direction?) ?

(24.10)

Example 2.2.3. 1. For any L one has the *trivial representation* $(k,0)$, where $V = k$ and R is the zero map.

2. For $a \in L$ define the linear map $\mathrm{ad}_L(a) = \mathrm{ad}(a) : L \to L$ by $\mathrm{ad}(a)b = [a,b]_L$. The *adjoint representation of* L is

$$\mathrm{ad} : L \to gl(L) \quad , \quad a \mapsto \mathrm{ad}_L(a) \ .$$

In this case, $V = L$ and $R = \mathrm{ad}$. We need to check that ad is a Lie algebra homomorphism: for all $a, b \in L$ and $v \in L$,

$$\begin{aligned}\big[\mathrm{ad}(a), \mathrm{ad}(b)\big]v &= \big(\mathrm{ad}(a)\,\mathrm{ad}(b) - \mathrm{ad}(b)\,\mathrm{ad}(a)\big)v \\ &= [a,[b,v]_L]_L - [b,[a,v]_L]_L \\ &= [a,[b,v]_L]_L + [b,[v,a]_L]_L = -[v,[a,b]_L]_L \\ &= [[a,b]_L, v]_L = \mathrm{ad}([a,b]_L)v \ .\end{aligned}$$

The derivations $\mathrm{Der}(L)$ on L (Example 1.4.3) are a Lie subalgebra of $gl(L)$, and $\mathrm{ad}(a)$ is a derivation on L (Why?). Thus the image of $\mathrm{ad} : L \to gl(L)$ actually lies in $\mathrm{Der}(L) \subset L$. ?

3. Let $V = \mathbb{C}[x,y] = \mathrm{span}_{\mathbb{C}}\{x^m y^n | n, m \in \mathbb{Z}_{\geq 0}\}$ be the polynomial ring over $\mathbb{C}$ in two variables. Then $R : sl(2,\mathbb{C}) \to gl(V)$,

$$R(e) = x\frac{\partial}{\partial y} \quad , \quad R(f) = y\frac{\partial}{\partial x} \quad , \quad R(h) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

defines an infinite-dimensional representation of $sl(2,\mathbb{C})$ on $\mathbb{C}[x,y]$. (Exercise) E

(Ex. 7)

Definition 2.2.4. Let (V,R) be a representation of L.

1. (V,R) is *faithful* if R is injective (i.e. $R(x) = R(y) \Rightarrow x = y$).

2. An *invariant subspace* or *subrepresentation* (or *submodule*) is a sub-vector space $U \subset V$ such that

$$R(a)u \in U \qquad \text{for all } a \in L, u \in U \ .$$

3. (V,R) is *irreducible* (or *simple*) if $V \neq \{0\}$ and if the only invariant subspaces are $\{0\}$ and V.

4. The *dual representation to* (V,R) is $(V,R)^* := (V^*, R^+)$, where V^* is the dual vector space, and $R^+(a) := -R^*(a) : V^* \to V^*$. (Why does this define a representation? What goes wrong if one omits the minus sign?) ?

5. Given two representations $V = (V, R)$ and $W = (W, S)$, an *intertwiner of representations* is a linear map $f : V \to W$ such that

$$\begin{array}{ccc} V & \xrightarrow{R(a)} & V \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{S(a)} & W \end{array}$$

commutes for all $a \in L$. As a formula: $S(a) \circ f = f \circ R(a)$. Write

$$\mathrm{Hom}_L(V, W) = \{f : V \to W \,|\, f \text{ is an intertwiner}\} \subset \mathrm{Hom}(V, W) ,$$

where $\mathrm{Hom}(V, W)$ denotes all k-linear maps $V \to W$.

Remark 2.2.5. Consider the adjoint representation (L, ad) of L. A subrepresentation $I \subset L$ is the same as an ideal in L (Why?). Thus, if L is a simple Lie algebra, (L, ad) is a simple representation of L. ?
For the converse there is a notational pitfall: The Lie algebra $L = k$ is not simple as a Lie algebra (since it is abelian), but (L, ad) is a simple representation of L.

Remark 2.2.6. Let $V = (V, R)$ and $W = (W, S)$ be representations of L.

1. Then $\mathrm{Hom}(V, W)$ is a representation of L. We describe it as an L-module instead: for all $a \in L$, $f \in \mathrm{Hom}(V, W)$,

$$a.f := S(a) \circ f - f \circ R(a) .$$

We need to check that this defines an L-module (Definition 2.2.1). First compute

$$\begin{aligned} &a.(b.f) = a.(S(b) \circ f - f \circ R(b)) \\ &= S(a) \circ S(b) \circ f - S(b) \circ f \circ R(a) + S(a) \circ f \circ R(b) + f \circ R(b) \circ R(a) \end{aligned}$$

Using this, we get

$$\begin{aligned} a.(b.f) - b.(a.f) &= [S(a), S(b)] \circ f + 0 + 0 + f \circ [R(b), R(a)] \\ &= S([a, b]) \circ f - f \circ R([a, b]) = [a, b].f \ . \end{aligned}$$

2. We say that L acts trivially on f if $a.f = 0$ for all $a \in L$. We have: (Why?) ?

$$L \text{ acts trivially on } f \ \Leftrightarrow\ f \text{ is an intertwiner} .$$

Thus $\mathrm{Hom}_L(V, W) \subset \mathrm{Hom}(V, W)$ is precisely the subspace of all f on which L acts trivially.

3. Let $f \in \mathrm{Hom}_L(V, W)$. As usual,

$$\ker f = \{v \in V \,|\, f(v) = 0\} \subset V \quad , \quad \mathrm{im}\, f = \{f(v) \,|\, v \in V\} \subset W \ .$$

Then $\ker f$ is a submodule of V and $\mathrm{im}\, f$ is a submodule of W. For example, if $v \in \ker f$, then

$$a.f(v) \overset{(*)}{=} f(a.v) = 0 \qquad \text{for all } a \in L \ ,$$

where $(*)$ uses that f is an intertwiner. Thus also $f(v) \in \ker(f)$ and so $\ker f$ is a submodule of V. (Can you rewrite the argument in terms of representations? Why is $\mathrm{im}\, f$ a submodule?) ?

Lemma 2.2.7. Let V, W be L-modules and $f : V \to W$ an intertwiner. Then:

1. If V is irreducible, then $f = 0$ or f is injective.
2. If W is irreducible, then $f = 0$ or f is surjective.
3. If V and W are irreducible, then $f = 0$ or f is bijective.

Proof. Parts 1 and 2 follow from Remark 2.2.6 (3). For example, part 1 follows from the fact that $\ker f \subset V$ is a submodule, and since V is irreducble, either $\ker f = \{0\}$ (i.e. f is injective) of $\ker f = V$ (i.e. $f = 0$). Part 2 is shown analogously (How?), and part 3 follows from parts 1 and 2. □ ?

Theorem 2.2.8 (Schur's Lemma)**.** Suppose k is algebraically closed (e.g. $k = \mathbb{C}$ but not $k = \mathbb{R}$). Let V be a finite-dimensional irreducible L-module and $f : V \to V$ an intertwiner. Then

$$f = \lambda \ \mathrm{id} \qquad \text{for some } \lambda \in k \ .$$

Proof. This follows from a linear algebra result:

> For k algebraically closed, a matrix $M \in \mathrm{Mat}(n, k)$ has at least one eigenvalue.

Let thus $v \in V$, $v \neq 0$ be an eigenvector of f with eigenvalue λ: $f(v) = \lambda v$. The linear map $g = f - \lambda\, \mathrm{id}$ is also an intertwiner $V \to V$. Since $v \in \ker g$, g is not bijective, and by Lemma 2.2.7 we must have $g = 0$. Hence $f = \lambda\, \mathrm{id}$. □

Remark 2.2.9. One cannot drop the conditions that k is algebraically closed or that V is finite-dimensional:

1. Let $L = \mathbb{R}$ be the 1-dimensional (automatically abelian) Lie algebra over $\mathbb{R}$. Write $u := 1 \in L$. This is a basis of L and $[u, u] = 0$. Set $R(u) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $(\mathbb{R}^2, R)$ is an irreducible representation of L. The linear map $f = R(u) : \mathbb{R}^2 \to \mathbb{R}^2$ is an intertwiner which is not proportional to the identity matrix. (Why is V irreducible? Is that still true if we use $k = \mathbb{C}$? Why is $R(u)$ an intertwiner? Can you work out $\mathrm{Hom}_L(V, V)$? This is an $\mathbb{R}$-algebra under composition, do you recognize it?) ?

2. (for info) An infinite-dimensional counter example with $k = \mathbb{C}$ can be constructed as follows: 1) Check that for an associative $\mathbb{C}$-algebra A and Lie algebra L over $\mathbb{C}$, the tensor product $A \otimes_{\mathbb{C}} L$ is again a Lie algebra. 2) Take $A = \mathbb{C}(t)$, the field of rational functions, and $L = sl(2, \mathbb{C})$. Check that the adjoint representation of $A \otimes_{\mathbb{C}} L$ is irreducible. 3) Check that for each $p \in C(t)$, the map $f(q \otimes a) = (pq) \otimes a$ is an intertwiner.

   These examples have to be a bit complicated as by Lemma 2.2.7, for V irreducible, $\mathrm{End}_L(V)$ is a division algebra. The only finite dimensional division algebra over $\mathbb{C}$ is $\mathbb{C}$. Thus in any counter-example, $\mathrm{End}_L(V)$ must be infinite dimensional over $\mathbb{C}$.

2.3 Direct sums and tensor products

■ From linear algebra: Let k be a field, and let V, W be k-vector spaces. From V, W one can construct their

$$\textit{direct sum } V \oplus W \quad \text{and} \quad \textit{tensor product } V \otimes_k W \ .$$

If the field k is clear, we write $V \otimes W$ instead of $V \otimes_k W$. (Why does one sometimes write $\otimes_k$ but not also $\oplus_k$? *Hint:* Think of $\mathbb{C}$ as a 1-dimensional $\mathbb{C}$ vector space and as a 2-dimensional $\mathbb{R}$-vector space. Compare $\mathbb{C} \oplus \mathbb{C}$ in both cases, as well as $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$.) ?

Remark 2.3.1. 1. If $\{v_i\}_{i \in I}$ and $\{w_j\}_{j \in J}$ are bases of V and W, respectively,

$$\{(v_i, 0)\}_{i \in I} \cup \{(0, w_j)\}_{j \in J} \text{ is a basis of } V \oplus W \ ,$$
$$\{v_i \otimes w_j\}_{(i,j) \in I \times J} \text{ is a basis of } V \otimes W \ .$$

If V, W are finite-dimensional, then

$$\dim(V \oplus W) = \dim V + \dim W \quad , \quad \dim(V \otimes W) = \dim V \cdot \dim W \ .$$

2. The tensor product is equipped with a bilinear map $V \times W \to V \otimes W$, $(v, w) \mapsto v \otimes w$. It has the following universal property:

   For every k-vector space X and every k-bilinear map $f : V \times W \to X$, there exists a unique k-linear map $\hat{f} : V \otimes W \to X$ such that $f = \hat{f} \circ \otimes$:

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[d, "\forall f"'] & V \otimes W \arrow[dl, dashed, "\exists! \hat{f}"] \\ X & \end{tikzcd}$$

   (Can you show that the pair $(V \otimes W, \otimes)$ in the universal property is unique up to unique isomorphisms? Can you show existence, e.g. by using bases? There is also a universal property for the direct sum, can you find it?) ?

3. Every vector $x \in V \oplus W$ can be *uniquely* written as $x = (v, w)$, $v \in V$, $w \in W$. We also write $x = v \oplus w$.

   An element of $V \otimes W$ of the form $v \otimes w$ is called a *pure tensor*. Not every vector $x \in V \otimes W$ is of the form $x = v \otimes w$, but x can always be written as a finite sum $x = \sum_i v_i \otimes w_j$. However, this representation is *not unique*.

   For example, for $V = W = \mathbb{R}^2$, $x = e_1 \otimes e_1 + e_2 \otimes e_2$ (e_i: standard basis) cannot be written as $x = v \otimes w$ (Why? *Hint:* Consider the bilinear map $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, $f(a, b) = g(v^\perp, a)b$, where $v^\perp$ is orthogonal to v.) And we have the identity (Why?) ?

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 2 \end{pmatrix} .$$

(27.10)

Definition 2.3.2. Let V, W be L-modules.

1. The *direct sum* $V \oplus W$ is the L-module with underlying vector space $V \oplus W$ and action, for $a \in L$,

$$a.(v \oplus w) := (a.v) \oplus (a.w) .$$

2. The *tensor product* $V \otimes W$ is the L-module with underlying vector space $V \otimes W$ and action, for $a \in L$,

$$a.(v \otimes w) := (a.v) \otimes w + v \otimes (a.w) .$$

Remark 2.3.3. Since an element in $V \otimes W$ can be represented in many different ways, it is not a priori clear that the above definition of $a.(v \otimes w)$ makes sense. To see that it does, it is best to use the universal property:

> The map $f_a : V \times W \to V \otimes W$, $f_a(v,w) = (a.v) \otimes w + v \otimes (a.w)$ is bilinear. Hence there is a unique linear map $\hat{f}_a : V \otimes W \to V \otimes W$ with the property $\hat{f}_a(v \otimes w) = f_a(v,w)$. For all $x \in V \otimes W$, define $a.x := \hat{f}_a(x)$.

An analogous argument has to be made whenever a linear map out of a tensor product is defined by its action on pure tensors.
So far we have defined a map $L \times (V \otimes W) \to V \otimes W$, $(a,x) \mapsto a.x$, which is linear in x. It remains to show that it is also linear in a, and that it satisfies the defining property of an L-module in Definition 2.2.1. (Details? Why does one not just set $a.(v \otimes w) = (a.v) \otimes (a.w)$?) ?

Example 2.3.4. Let $B : L \otimes L \to L$ be defined by $B(a \otimes b) = [a,b]_L$. (Why is this well-defined?) ?

Claim: B is an intertwiner of L-modules.

⌈ We need to check that for all $a, v, w \in L$ we have $a.B(v \otimes w) = B(a.(v \otimes w))$. Indeed,

$$a.B(v \otimes w) = a.[v,w] = [a,[v,w]]$$

and

$$\begin{aligned} B(a.(v \otimes w)) &= B((a.v) \otimes w + v \otimes (a.w)) \\ &= B([a,v] \otimes w + v \otimes [a,w]) = [[a,v],w] + [v,[a,w]] \ . \end{aligned}$$

Equality follows from the Jacobi-identity. ⌋

Definition 2.3.5. For Lie algebras L, L' over k, the *direct sum of L and L'* is the Lie algebra with underlying vector space $L \oplus L'$ and bracket

$$\big[a \oplus a' \, , \, b \oplus b'\big] := [a,a']_L \, \oplus \, [b,b']_{L'} \ .$$

Remark 2.3.6.

1. We identify L and L' with their summands in $L \oplus L'$: $L \equiv L \oplus 0 \subset L \oplus L'$, etc. Then L and L' commute in $L \oplus L'$ in the sense that $[L, L'] = \{0\}$.

2. Conversely, let $A, B \subset L$ be Lie subalgebras such that $[A,B] = \{0\}$ and $L = A \oplus B$ as vector spaces. Then $L = A \oplus B$ as Lie algebras.

3. Let $L = A \oplus B$ as vector spaces. Then $L = A \oplus B$ as Lie algebras if and only if A and B are ideals in L. (Why?) ?

Example 2.3.7. Recall the Lie algebra $u(n)$ over $\mathbb{R}$ from Example 1.5.7. Note that $u(1)$ is a 1-dimensional abelian Lie algebra over $\mathbb{R}$ (given by 1×1-matrices of the form $(i\lambda)$, $\lambda \in \mathbb{R}$).

Claim: $u(n) \cong u(1) \oplus su(n)$ as Lie algebras.

⌈ The linear map $f : u(n) \to u(1) \oplus su(n)$,

$$f(A) = \mathrm{tr}(A) \oplus \big(A - \tfrac{1}{n}\,\mathrm{tr}(A)\mathbb{1}\big) \ .$$

is an isomorphism of Lie algebras. (Why?) ⌋ ?

For info: The Lie groups $U(1) \times SU(n)$ and $U(n)$ are not isomorphic as Lie groups. Indeed, they are not even isomorphic as groups: $U(1) \times SU(n)$ has centre $U(1) \times \mathbb{Z}_n$ and $U(n)$ has centre $U(1)$, the latter group is divisible, the former is not. But there is a surjective smooth group homomorphism $\phi : U(1) \times SU(n) \to U(n)$, $\phi(\lambda, M) = \lambda M$ with finite kernel

$$\ker(\phi) = \{(e^{2\pi i k/n}, e^{-2\pi i k/n}\mathbb{1}) \,|\, k \in \mathbb{Z}\} \cong \mathbb{Z}_n \ .$$

which turns $U(1) \times SU(n)$ into a finite (connected) covering of $U(n)$. (The universal cover of both groups is $\mathbb{R} \times SU(n)$.)

However, $U(1) \times SU(n)$ and $U(n)$ are diffeomorphic as manifolds. For example, one can choose $f : U(1) \times SU(n) \to U(n)$ and $g : U(n) \to U(1) \times SU(n)$ to be

$$f(\lambda, M) = \mathrm{diag}(\lambda, 1, \dots, 1)M \quad , \quad g(A) = (\det A, A/\det(A)) \ ,$$

which are smooth and mutually inverse to each other.

3 Nilpotent and solvable Lie algebras

In this section, k denotes a field and L a Lie algebra over k.

3.1 Nilpotent Lie algebras and Engel's Theorem

Definition 3.1.1. 1. The *descending central series of L* is defined recursively by

$$C^0 L = L \quad , \quad C^{i+1} L = [L, C^i L] \text{ for } i \geq 0 \ .$$

L is called *nilpotent* if $C^n L = \{0\}$ for some n.

2. An element $x \in L$ is called *ad-nilpotent* if $\mathrm{ad}(x) : L \to L$ is a nilpotent endomorphism, i.e. $\mathrm{ad}(x)^N = 0$ for some N.

Remark 3.1.2. 1. *Claim:* C^iL is an ideal in L

⌈ Let $x \in L$ and $a \in C^iL$. We proceed by induction. For $i = 0$ there is nothing to show. If $i > 0$, a can be written as a linear combination of terms $[y, b]$ with $y \in L$ and $b \in C^{i-1}$. By induction hypothesis, $C^{i-1}L$ is an ideal in L, and so $[y, b] \in C^{i-1}L$. Altogether, $[x, a]$ is a linear combination of terms $[x, [y, b]] \in [L, C^{i-1}L] = C^iL$. ⌋

2. Part 1 implies that $C^{i+1}L \subset C^iL$, and that C^iL is a Lie subalgebra and $C^{i+1}L \subset C^iL$ is an ideal.

   Claim: The Lie algebra $C^iL/C^{i+1}L$ is abelian.

   (Why?) ?

3. Let $x, y \in L$. Then $\mathrm{ad}(x)^i y \in C^iL$. In particular, if L is nilpotent every element of L is ad-nilpotent.

Theorem 3.1.3 (Engel's Theorem)**.** Let L be a finite-dimensional Lie algebra over a field k. If all $x \in L$ are ad-nilpotent, then L is nilpotent.

The proof of Theorem 3.1.3 requires some preparation.

Remark 3.1.4. 1. For infinite-dimensional L, the conclusion of the theorem does in general not hold. (Exercise) E (Ex. 11)

2. It is not enough to check that all elements of a given basis of L are ad-nilpotent to conclude that L is nilpotent. For example $sl(2, \mathbb{C})$ is not nilpotent (Why?), but the three basis vectors ?

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad x = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

are all ad-nilpotent. (For e and f this is immediate from the brackets given in the $sl(2, k)$ example in Section 2.1, but x needs to be checked. Can you give an $A \in sl(2, \mathbb{C})$ that is not ad-nilpotent?) ?

Main example

Define the *upper triangular matrices*

$$T(n) = T(n, k) = \mathrm{span}_k\{\, E_{ij} \,|\, 1 \le i \le j \le n \,\} \ ,$$

the *strictly upper triangular matrices*

$$N(n) = N(n,k) = \mathrm{span}_k\{\, E_{ij} \,|\, 1 \le i < j \le n \,\}\ ,$$

and the *diagonal matrices*

$$D(n) = D(n,k) = \mathrm{span}_k\{\, E_{ii} \,|\, 1 \le i \le n \,\}\ .$$

Then $T(n) = D(n) \oplus N(n)$ as vector spaces, but not as Lie algebras. For $a \in \mathbb{Z}_{\ge 0}$ write

$$T_a = \mathrm{span}_k\{\, E_{ij} \,|\, i + a \le j \,\}\ ,$$

so that $T_0 = T(n)$, $T_1 = N(n)$, $T_a = \{0\}$ for $a \ge n$, and $T_a \subset T_b$ for $a \ge b$.

Claim: $[T_a, T_b] \subset T_{a+b}$, with "=" holding for $a + b \ge 1$.

⌈ The inclusion follows from the observation that $[E_{ij}, E_{xy}] = \delta_{j,x}E_{iy} - \delta_{y,i}E_{xj}$, together with $i + a \le j$ and $x + b \le y$.

Let $E_{uv} \in T_{a+b}$, i.e. $u + a + b \le v$. We need to write E_{uv} as (a sum of) commutators of matrices in T_a and T_b.

Note that $E_{u,u+a} \in T_a$ and $E_{u+a,v} \in T_b$, and that

$$[E_{u,u+a}, E_{u+a,v}] = E_{u,u+a}E_{u+a,v} - E_{u+a,v}E_{u,u+a} = E_{uv} - \delta_{v,u}E_{u+a,u+a}\ .$$

But $u + a + b \le v$ and $a + b \ge 1$, and so $u \ne v$. ⌋

We have the following properties for $T(n)$ and $N(n)$:

1. $[T(n), T(n)] = N(n)$ and $[T(n), N(n)] = N(n)$. (The second equality is a special case of the claim, why does the first one hold?) ?

2. Part 1 shows that $T(n)$ is not nilpotent, since

$$C^i T(n) = \begin{cases} T(n) & ;\ i = 0 \\ N(n) & ;\ i > 0 \end{cases}\ .$$

   On the other hand, the above claim shows that $N(n)$ is nilpotent with (3.11.)

$$C^i N(n) = T_{i+1}$$

Preparation for the proof of Engel's Theorem

Proposition 3.1.5. 1. Suppose L is nilpotent. Then

(a) If $K \subset L$ is a subalgebra, then K is nilpotent.

(b) If $\varphi : L \to L'$ is a Lie algebra homomorphism, then $\operatorname{im}\varphi \subset L'$ is nilpotent.

(c) If $L \neq \{0\}$, then $Z(L) \neq \{0\}$.

2. $L/Z(L)$ is nilpotent iff L is nilpotent.

Proof. 1a) $K \subset L \Rightarrow [K, K] \subset [L, L] \Rightarrow \cdots \Rightarrow C^i K \subset C^i L$ for all i.

1b) $[\varphi(L), \varphi(L)] = \varphi([L, L])$ and inductively $C^i(\varphi(L)) = \varphi(C^i L)$.

1c) Suppose $C^n L = \{0\}$ but $C^{n-1}L \neq \{0\}$. Then $[L, C^{n-1}L] = \{0\}$ and so $C^{n-1}L \subset Z(L)$.

2) Suppose L is nilpotent. By part 1 (b), the image of $\pi : L \to L/Z(L)$ is nilpotent, but π is surjective.

Conversely, suppose $L/Z(L)$ is nilpotent. There there is an N such that for all $x_1, \dots, x_N, y \in L$, in $L/Z(L)$ we have

$$\operatorname{ad}(\pi(x_1))\operatorname{ad}(\pi(x_2))\cdots\operatorname{ad}(\pi(x_N))\pi(y) = 0 \ .$$

Equivalently $z := \operatorname{ad}(x_1)\operatorname{ad}(x_2)\cdots\operatorname{ad}(x_N)y \in Z(L)$ (Why is this equivalent?). But then $[x_0, z] = 0$ for all $x_0 \in L$, and so $C^{N+1}L = 0$. ?

□

Definition 3.1.6. For a sub-vector space $K \subset L$ the *normaliser of K in L* is

$$N_L(K) = \{x \in L \,|\, [x, a] \in K \text{ for all } a \in K\} \ .$$

Remark 3.1.7. $[N_L(K), K] \subset K$, and $N_L(K)$ is a Lie subalgebra of L (Why?). If K is itself a Lie subalgebra (rather than just a sub-vector space), then $K \subset N_L(K)$ and in fact K is an ideal in $N_L(K)$. ?

■ Recall that for a vector space V, $gl(V) = \operatorname{End}(V)$, with $\operatorname{End}(V)$ the associative algebra of linear maps $V \to V$, and $gl(V)$ the corresponding Lie algebra with commutator as Lie bracket. $x \in gl(V)$ is *nilpotent*, if $x^N = x \circ x \circ \cdots \circ x = 0$ for some N.

Lemma 3.1.8. Let V be a k-vector space. If $x \in gl(V)$ is nilpotent, then x is also ad-nilpotent.

(The converse if false. Can you give a counter example?) ?

Proof. We need to show that the linear map $\mathrm{ad}(x) : gl(V) \to gl(V)$ is nilpotent. Define $\lambda_x, \rho_x : gl(V) \to gl(V)$ as

$$\lambda_x(a) = x \circ a \quad , \quad \rho_x(a) = a \circ x \ .$$

Then $\mathrm{ad}(x) = \lambda_x - \rho_x$ and $\lambda_x \circ \rho_x = \rho_x \circ \lambda_x$ (Why?). Suppose $x^N = 0$ for some N. Then also $(\lambda_x)^N = 0 = (\rho_x)^N$. We have ?

$$\mathrm{ad}(x)^{2N} = (\lambda_x - \rho_x)^{2N} \overset{(*)}{=} \sum_{j=0}^{2N} \binom{2N}{j} \lambda_x^j \, \rho_x^{2N-j} = 0 \ ,$$

where $(*)$ uses that λ_x and ρ_x commute. □

Theorem 3.1.9. Let $V \neq \{0\}$ be a finite-dimensional k-vector space, and let $L \subset gl(V)$ be a Lie subalgebra. If all $x \in L$ are nilpotent as maps $V \to V$, then there is a $v \in V$, $v \neq 0$, such that $Lv = \{0\}$, i.e. $\forall x \in L : x \circ v = 0$.

(We cannot just demand that L is a linear subspace of $gl(V)$. Can you give a counter example?) ?

Proof. We use induction on the dimension of L. Let $A(n)$ denote the statement of the theorem with the addtional assumption that $\dim L \leq n$.

Base case $n = 1$: For $\dim L = 0$ there is nothing to do. Suppose $\dim L = 1$ and let $x \in L$ be non-zero. By assumption, $x^N = 0$ for some N, and so x cannot be bijective. Hence $\ker x \neq \{0\}$.

Induction step $A(n-1) \Rightarrow A(n)$: We may assume that $\dim L = n$.

Claim 1: Let $K \subsetneq L$ be a Lie subalgebra. Then $K \subsetneq N_L(K)$.

$\lceil$ K acts on K and on L via the adjoint action, and (K, ad) is a subrepresentation (for the K-action) of (L, ad). Hence we obtain a well-defined action of K on L/K, i.e. a Lie algebra homomorphism $R : K \to gl(L/K)$ (Details?). Write $\overline{K} = \mathrm{im}(R) \subset gl(L/K)$. ?

Let $x \in K$. By assumption, $x : L \to L$ is nilpotent, and by Lemma 3.1.8, $\mathrm{ad}(x) : L \to L$ is nilpotent. But then each $a \in \overline{K}$ is a nilpotent map $L/K \to L/K$ (Why?). ?

Since $\dim \overline{K} \leq \dim K < \dim L = n$ we may apply $A(n-1)$ to $V = L/K$ and $\overline{K} \subset gl(L/K)$:

$$\exists \overline{v} \in L/K \ : \ \overline{v} \neq 0 \text{ and } \overline{K}\overline{v} = \{0\} \ .$$

In terms of L, this means there is $v \in L$, $v \notin K$ such that $[x, v] \in K$ for all $x \in K$. By Definition 3.1.6, $v \in N_L(K)$. Since anyway $K \subset N_L(K)$ (Remark 3.1.7) and $v \notin K$, we have $K \subsetneq N_L(K)$. ⌋

Claim 2: Let $K \subsetneq L$ be a maximal Lie subalgebra. Then K is an ideal in L.

⌈ $N_L(K)$ is a Lie subalgebra (Remark 3.1.7), and by Claim 1 we have $K \subsetneq N_L(K)$. But by assumption K is maximal, and so we must have $N_L(K) = L$, i.e. K is an ideal in L. ⌋

Claim 3: Let K be as in Claim 2. Then $\dim(L/K) = 1$.

⌈ Suppose $\dim L/K > 1$. By Claim 2, L/K is a Lie algebra. Let $\pi : L \to L/K$ be the canonical projection. Let $\{0\} \subsetneq \overline{U} \subsetneq L/K$ be a one-dimensional subspace (and so in particular a Lie subalgebra). Then $U := \pi^{-1}(\overline{U})$ is a Lie subalgebra of L with $K \subsetneq U \subsetneq L$, in contradiction to maximality of K. ⌋

Claim 4: Let K be as in Claim 2 and set $W := \{v \in V \,|\, Kv = 0\}$. Then there is a non-zero $w \in W$ such that $Lw = \{0\}$.

⌈ By Claim 3 we can find $z \in L$ such that $L = K \oplus kz$. Let $v \in W$. Then also $zv \in W$ as for all $x \in K$,

$$xzv = \underbrace{[x, z]}_{\in K} v + z \underbrace{xv}_{=0} = 0 \ ,$$

where we used that K is an ideal in L (Claim 2). By assumption, all elements of L are nilpotent on V, so in particular, z is nilpotent on W.

Applying $A(n-1)$ to K and V shows that $W \neq \{0\}$. Since z is nilpotent on W, there is a non-zero $w \in W$ such that $zw = 0$. Since by definition $Kw = \{0\}$ it follows that $Lw = (K \oplus kz)w = \{0\}$. ⌋

A maximal proper Lie algebra always exists (Why?). Hence Claim 4 completes the induction step. □ ?

Definition 3.1.10. Let V be a finite-dimensional k-vector space and set $n = \dim(V)$. A *flag* (V_i) *in* V is a collection of subspaces

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V \ ,$$

such that $\dim V_i = i$.

Corollary 3.1.11. Let V, L be as in Theorem 3.1.9. Then there exists a flag (V_i) in V such that

$$\forall\, x \in L,\ i = 1, \dots, \dim V \ :\ xV_i \subset V_{i-1}\ .$$

Proof. Suppose we have already constructed the flag up to some $i \geq 0$: $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_i$.

If $V_i = V$ we are done. Else consider the quotient vector space $\overline{V} = V/V_i$ with canonical projection $\pi : V \to \overline{V}$.

For each $x \in L$ there is a unique $\overline{x} : \overline{V} \to \overline{V}$ such that $\pi \circ x = \overline{x} \circ \pi$, and the map $\varphi : L \to gl(\overline{V})$, $x \mapsto \overline{x}$ is a Lie algebra homomorphism (Why?). Set $\overline{L} := \mathrm{im}(\varphi)$. By Proposition 3.1.5 (1b), $\overline{L}$ is nilpotent. Theorem 3.1.9, applied to $\overline{V}$ and $\overline{L}$ gives $v \in \overline{V}$, $v \neq 0$, such that $\overline{L}v = \{0\}$. Take $V_{i+1} = \pi^{-1}(kv)$. (Why does V_{i+1} have the required properties to give the next part of the flag?) ∎ ?

?

Remark 3.1.12. Let V, L be as in Theorem 3.1.9, and let (V_i) be a flag as in the above corollary. Let $e_1, \dots, e_n$ be a basis of V such that $e_1, \dots, e_i$ is a basis of V_i. The basis defines an isomorphism $\phi : V \to k^n$, and hence an isomorphism of Lie algebras

$$gl(V) \to gl(n, k) \quad , \quad f \mapsto \phi \circ f \circ \phi^{-1}\ .$$

The image of L under this isomorphism lies in $N(n, k)$. (Why?) ?

(7.11)

Proof of Engel's Theorem

Recall the statement of Engel's Theorem (Theorem 3.1.3):

$$(L \text{ Lie alg.},\ \dim L < \infty\ ,\quad \forall x \in L : x \text{ ad-nilpotent}) \ \Rightarrow\ L \text{ nilpotent}$$

Proof of Theorem 3.1.3. We again proceed by induction. Let thus $A(n)$ be the statement of Engel's Theorem plus the assumption that $\dim L = n$.

Base case $n = 1$: Clear.

Induction step $A(n-1) \Rightarrow A(n)$:

Claim: $Z(L) \neq \{0\}$.

⌈ Consider the adjoint representation $\mathrm{ad} : L \to gl(L)$ and write $\widetilde{L} := \mathrm{im}(\mathrm{ad}) \subset gl(L)$. Since each $x \in L$ is ad-nilpotent, by definition each $u \in \widetilde{L}$ is a nilpotent endomorphism of L. By Theorem 3.1.9 there is $v \in L$, $v \neq 0$, such that $\widetilde{L}v = 0$. But this is equivalent to $[x, v] = 0$ for all $x \in L$. ⌋

Let $K = L/Z(L)$. As all $x \in L$ are ad-nilpotent, also all $a \in K$ are ad-nilpotent (Why?). By the above claim, $\dim K < n$. Thus the induction hypothesis $A(n-1)$ applies to K, showing that K is nilpotent. By Proposition 3.1.5 (2), L is nilpotent. ☐

3.2 Solvable Lie algebras and Lie's Theorem

Definition 3.2.1. The *derived series of* L is defined recursively by

$$D^0 L = L \quad , \quad D^{i+1}L = [D^i L, D^i L] \text{ for } i \geq 0 \ .$$

L is called *solvable* if $D^n L = \{0\}$ for some n.

Remark 3.2.2.
1. $T(n,k)$ is solvable, as by the main example in Section 3.1: $D^1T(n,k) = [T(n,k), T(n,k)] = [T_0, T_0] = T_1$, $D^2T(n,k) = [T_1, T_1] = T_2$, $D^3T(n,k) = [T_2, T_2] = T_4$, etc.
2. Nilpotent implies solvable, but in general solvable does not imply nilpotent. (Why?)

Theorem 3.2.3 (Lie's Theorem)**.** Let k be algebraically closed and of characteristic zero. Let V be a finite-dimensional k-vector space and $L \subset gl(V)$ a solvable subalgebra. Then L stabilises a flag in V.

■ That L stabilises a flag in V means that there is a flag $(V_i)_{i=0,\dots,\dim V}$ of V such that for all $x \in L$ and all i we have $xV_i \subset V_i$.

In terms of the basis chosen in Remark 3.1.12 (i.e. $e_1, \dots, e_i$ is a basis of V_i), each $x \in L$ is represented by an upper triangular matrix. This gives a Lie algebra homomorphism $L \to T(n,k)$.

The proof of Lie's Theorem requires some preparation

Preparation for the proof of Lie's Theorem

Proposition 3.2.4.
1. Suppose L is solvable. If $K \subset L$ is a Lie subalgebra, then K is solvable. If $\varphi : L \to L'$ is a Lie algebra homomorphism, then $\mathrm{im}(\varphi) \subset L'$ is solvable.
2. Let $I \subset L$ be an ideal. If I and L/I are solvable, then L is solvable.
3. If $I, J \subset L$ are ideals and I, J are solvable, then $I + J$ is solvable.

Remark 3.2.5.
1. Let $0 \to K \to L \to Q \to 0$ be a short exact sequence of Lie algebras (i.e. image = kernel and the arrows are Lie algebra

homomorphism; "short" means that there are three non-trivial terms). Combining parts 1 and 2 of the proposition shows

$$L \text{ solvable} \quad \Leftrightarrow \quad K, Q \text{ solvable} .$$

2. Part 3 shows that each finite dimensional Lie algebra contains a unique maximal solvable ideal. For suppose $I, J \subset L$ are solvable and maximal in the sense that there are no $I' \supsetneq I$ and $J' \supsetneq J$ which are solvable. Since $I, J \subset I + J$ and $I + J$ is solvable, we must have $I = I + J = J$. (What goes wrong if we drop "finite-dimensional"?) ?

Proof of Proposition 3.2.4. Part 1: Similar to Proposition 3.1.5 (1), in particular $\varphi(D^i L) = D^i(\varphi(L))$. (Details?) ?

Part 2: We have the short exact sequence, with $Q = L/I$,

$$0 \to I \xrightarrow{\iota} L \xrightarrow{\pi} Q \to 0 .$$

Since Q is solvable, there is an m such that $D^m Q = \{0\}$. Then

$$\pi(D^m L) = D^m(\pi(L)) = D^m Q = \{0\} ,$$

and so $D^m L \subset I$. Since I is solvable, there is n such that $D^n I = \{0\}$ and so also $D^{m+n} L = \{0\}$. (Why does the corresponding argument not work for nilpotent Lie algebras?) ?

Part 3: Consider the short exact sequence

$$0 \to J \xrightarrow{\iota} I + J \xrightarrow{\pi} (I + J)/J \to 0 .$$

The projection $I \to (I + J)/J$, $x \mapsto [x]$ is a surjective Lie algebra homomorphism, and since I is solvable, by part 1 also $(I + J)/J$ is solvable. Since also J is solvable, but part 2, so is $I + J$. $\square$

Definition 3.2.6. Let L be a finite-dimensional Lie algebra over k.

1. The *radical* $\operatorname{Rad} L$ *of* L is the (unique) maximal solvable ideal of L.

2. L is called *semisimple* if $\operatorname{Rad} L = \{0\}$.

■ A simple finite-dimensional Lie algebra is in particular semisimple. (Why? ? *Hint:* Either $\operatorname{Rad} L = \{0\}$ or $\operatorname{Rad} L = L$. But also $[L, L]$ is an ideal, and so $[L, L] = \{0\}$ or $[L, L] = L$.)

Finite-dimensional semisimple Lie algebras over $\mathbb{C}$ are the topic of Sections 5 and 6.

Proposition 3.2.7. Let k be algebraically closed and of characteristic zero. Let V be a finite-dimensional non-zero k-vector space, and $L \subset gl(V)$ a solvable Lie subalgebra. Then V contains a common eigenvector for all $x \in L$.

■ Here, "common eigenvector" means that there is $v \in V$, $v \neq 0$ and a function $\lambda : L \to k$, such that

$$\forall x \in L \ : \ xv \ = \ \lambda(x)\, v \ .$$

Then λ is automatically k-linear. (Why?) ?

The proof of Proposition 3.2.7 needs the following lemma.

Lemma 3.2.8. Let k be of characteristic zero, L a Lie algebra over k, and $H \subset L$ an ideal. Let M be a finite-dimensional L-module. Suppose there is $v \in M$, $v \neq 0$, and $\lambda : H \to k$ such that

$$h.v = \lambda(h)v \quad \text{for all} \ \ x \in L, h \in H \ .$$

Then $\lambda([x,h]) = 0$ for all $x \in L$, $h \in H$.

Proof. Let $x \in L$ be given and write $R(x) : M \to M$ for the action of x on M. Let $n \geq 0$ be maximal such that

$$\{\, v\,,\ R(x)v\,,\ \dots\ ,\ R(x)^n v\,\} \subset M$$

is linearly independent. Abbreviate $v_i = R(x)^i v$ and set $U_i = \mathrm{span}_k\{v_0, v_1, \dots, v_i\}$ so that $\dim U_i = i + 1$.

Claim: For all $h \in H$ and all i we have $h.v_i = \lambda(h)v_i + w$ for some $w \in U_{i-1}$ ($w = 0$ for $i = 0$).

⌈ We proceed by induction on i. For $i = 0$ the statement holds by assumption. Suppose the statement holds for a given i. Then

$$\begin{aligned} h.v_{i+1} = h.(x.v_i) &\overset{(2)}{=} [h,x].v_i + x.(h.v_i) \\ &\overset{(3)}{=} w + x.\big(\lambda(h)v_i + w'\big) \overset{(4)}{=} \lambda(h)v_{i+1} + w'' \ . \end{aligned}$$

In step (2) we use $[h,x] \in H$ to conclude $w := [h,x].v_i \in U_i$ by induction hypothesis. Step (3) is the induction hypothesis once more, with $w' \in U_{i-1}$ In step (4) we set $w'' = w + x.w' \in U_i$. ⌋

By the claim, in the basis $\{v_j\}_{j=0,\dots,n}$, $R(h)$ takes the form

$$R(h) = \begin{pmatrix} \lambda(h) & & * \\ & \ddots & \\ 0 & & \lambda(h) \end{pmatrix}$$

Thus $\mathrm{tr}_{U_n} R(h) = (n+1)\lambda(h)$, and so

$$(n+1)\lambda([x,h]) = \mathrm{tr}_{U_n} R([x,h]) = \mathrm{tr}_{U_n} \big(R(x)R(h) - R(h)R(x)\big) = 0 \ .$$

Since $\mathrm{char}(k) = 0$, this implies $\lambda([x,h]) = 0$.

(Where did we use the fact that the n in U_n is defined by the maximal linear independence property above? Can we just omit this demand on n? *Hint:* To get zero in the last equality above, we use cyclicity of the trace, and we need that $R(x)$ and $R(h)$ restrict to maps $U_n \to U_n$.) ? $\square$

Proof of Proposition 3.2.7. We proceed by induction on $\dim L$. Let thus $A(n)$ be the statement of the proposition, under the additional assumption that $\dim L \leq n$.

Base case $n = 0$: Clear.

Induction step $A(n-1) \Rightarrow A(n)$: We may assume that $\dim L = n \geq 1$.

Claim 1: L contains an ideal K of codimension 1.

> ⌈ Since L is solvable, $D^1L = [L,L] \subsetneq L$. The quotient $Q := L/[L,L]$ is non-zero and abelian. Hence there exists a sub-vector space $U \subset Q$ of codimension 1. As Q is abelian, U is an ideal of Q. Write $\pi : L \to Q$ for the canonical projection. The preimage $K := \pi^{-1}(U)$ is of codimension 1 in L and is an ideal. (Why?) ⌋ ?

Since $K \subset gl(V)$ is solvable (Proposition 3.2.4 (1)) and of dimension $n-1$, $A(n-1)$ applies and V contains a common eigenvector $v \neq 0$ for K, of eigenvalue function $\lambda : K \to k$. Let W_λ be the common eigenspace for K with eigenvalue function λ:

$$W_\lambda = \{ w \in W \,|\, \forall x \in K : xw = \lambda(x)w \} \ .$$

Since $v \in W_\lambda$, in particular W_λ is non-zero.

Claim 2: W_λ is an invariant subspace for L, i.e. $\forall a \in L : aW_\lambda \subset W_\lambda$.

> ⌈ Let $a \in L$ be given. We need to check that for all $w \in W_\lambda$ also $aw \in W_\lambda$, i.e. that for all $x \in K$ we have $xaw = \lambda(x)aw$. To see this, first note
>
> $$xaw = [x,a]w + axw \overset{(*)}{=} \lambda([x,a])w + \lambda(x)aw \ ,$$
>
> where in $(*)$ we used $[a,x] \in K$ (K is an ideal by claim 1). By Lemma 3.2.8 (with $H \rightsquigarrow K$, $M \rightsquigarrow W_\lambda$), $\lambda([x,a]) = 0$, and so indeed $x(aw) = \lambda(x)aw$. ⌋

Pick $z \in L \backslash K$. As K has codimension 1, we have $L = K \oplus kz$. By Claim 2, z restricts to a map $z : W_\lambda \to W_\lambda$. As W_λ is non-zero (and finite-dimensional, and the field is algebraically closed), it contains an eigenvector $w \in W_\lambda$ for z: $zw = \mu w$ for some $\mu \in k$. Then $w \in V$ is a common eigenvector for all $x \in L$. (Why? What is the eigenvalue function $\lambda' : L \to k$? Where in the proof did we use that k is characteristic zero?) ? □ (10.11)

Proof of Lie's Theorem

Recall the statement of Lie's Theorem (Theorem 3.2.3):

$$\begin{aligned} &\big(\overline{k} = k\,,\ \mathrm{char}(k) = 0\,,\ \dim V < \infty\,,\ L \subset gl(V) \text{ solvable}\big) \\ &\Rightarrow \quad L \text{ stabilises a flag in } V \end{aligned}$$

Proof of Theorem 3.2.3. If $\dim V = 0$, there is nothing to show. Suppose $\dim V \geq 1$. By Proposition 3.2.7 there is a common eigenvalue $v \in V$ for L. Take $V_1 = kv$. Then repeat the argument for $\overline{V} := V/V_1$. (Details? *Hint:* $V_1 \subset V$ is an L-submodule, so $\overline{V}$ is again an L-module; consider the image of L in $gl(\overline{V})$.) ? □

Proposition 3.2.9. Let k be an algebraically closed field of characteristic zero, and let L be a finite-dimensional solvable Lie algebra over k. Then:

1. There is a flag $(L_i)_{i=0,\dots,\dim L}$ in L such that each L_i is an ideal of L.

2. $[L, L]$ is nilpotent.

Proof. Part 1: Consider the adjoint representation $\mathrm{ad} : L \to gl(L)$ and set $\overline{L} := \mathrm{im}\,\mathrm{ad}$. By Proposition 3.2.4 (1), $\overline{L}$ is solvable. By Theorem 3.2.3, $\overline{L}$ stabilises a flag (L_i) in L. Each L_i in this flag is an ideal of L. (Why?) ?

Part 2: Let (L_i) be the flag from part 1. For each $u \in [L, L]$ we have $\mathrm{ad}(u)L_i \subset L_{i-1}$ (Exercise). Thus all $\mathrm{ad}(u)$ are nilpotent, and by Engel's Theorem (Theorem 3.1.3), $[L, L]$ is nilpotent. E □ (Ex. 15)

3.3 Cartan's criterion for solvability

Theorem 3.3.1 (Cartan's criterion)**.** Let k be algebraically closed and of characteristic zero. Let V be a finite-dimensional k-vector space and $L \subset gl(V)$ a subalgebra. If $\mathrm{tr}_V(xy) = 0$ for all $x \in [L, L]$ and $y \in L$, then L is solvable.

The proof will be given at the end of this section.

Remark 3.3.2. The converse statement (L solvable $\Rightarrow \mathrm{tr}_V(xy) = 0$ for all $x \in [L, L]$ and $y \in L$) can be obtained from Lie's Theorem. (Exercise) E

(Ex. 15)

Definition 3.3.3. Let L be a finite-dimensional Lie algebra. The *Killing form* κ_L (or just κ) of L is the bilinear map $\kappa_L : L \times L \to k$ given by

$$\kappa_L(x, y) := \mathrm{tr}_L\left(\mathrm{ad}(x)\,\mathrm{ad}(y)\right) \qquad \text{for all } \; x, y \in L \ .$$

Example 3.3.4.

1. If $L = N(n, k)$, then $\kappa_L(x, y) = 0$ for all $x, y \in N(n, k)$. For $L = gl(n, k)$, the Killing form is non-degenerate (i.e. for all $x \in gl(n, k)$, $x \neq 0$, there is $y \in gl(n, k)$ such that $\kappa_L(x, y) \neq 0$). (Why?) ?

2. Let $L = sl(2, k)$. In the basis e, h, f used above Proposition 2.1.9, we have
$$[h, e] = 2e \quad , \quad [h, f] = -2f \quad , \quad [e, f] = h$$
and so
$$\mathrm{ad}(e) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} , \ \mathrm{ad}(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} , \ \mathrm{ad}(f) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} .$$
This gives, for example,
$$\kappa(e, h) = 0 \ , \quad \kappa(h, h) = 8 \ , \quad \kappa(e, f) = 4 \ .$$

Corollary 3.3.5 (to Cartan's criterion)**.** Let k be algebraically closed and of characteristic zero, and let L be a finite-dimensional Lie algebra over k. The following are equivalent:

1. L is solvable.

2. $\kappa_L(x, y) = 0$ for all $x \in [L, L]$, $y \in L$.

(Is the corollary still true if in 2. we demand instead that $x, y \in L$?) ?

Proof. Consider the adjoint representation $\mathrm{ad} : L \to gl(L)$ and set $\overline{L} = \mathrm{im}(\mathrm{ad}) \subset gl(L)$. Then $\overline{L}$ is solvable by Proposition 3.2.4 (1).

$1 \Rightarrow 2$: This follows from Remark 3.3.2. (Details?) ?

$2 \Rightarrow 1$: By Cartan's criterion (Theorem 3.3.1) $\overline{L}$ is solvable. We have the short exact sequence (Why?) ?

$$0 \to Z(L) \to L \xrightarrow{ad} \overline{L} \to 0 \ .$$

Since $\overline{L}$ is solvable, and $Z(L)$ is always solvable, by Remark 3.2.5 (1), L is solvable. □

Remark 3.3.6. Let L be a finite-dimensional Lie algebra.

1. The Killing form of L is *symmetric* and *invariant*: For all $x, y, z \in L$ we have (Why does the invariance property hold?) ?

$$\kappa_L(x, y) = \kappa_L(y, x) \quad , \qquad \kappa_L([x, y], z) = \kappa_L(x, [y, z]) \ .$$

2. Let $I \subset L$ be an ideal. Then I is in particular a Lie subalgebra and we can consider its Killing form κ_I.

   Claim: For all $a, b \in I$ we have $\kappa_I(a, b) = \kappa_L(a, b)$.

   ⌈ Pick a complement of I in L, i.e. a sub-vector space $V \subset L$ such that $L = I \oplus V$ (as vector spaces, not as Lie algebras). Then $\mathrm{ad}(a)\,\mathrm{ad}(b) : I \oplus V \to I$ (as I is an ideal), and the component $V \to I$ does not contribute to the trace. ⌋

For a subset $V \subset L$ write $V^\perp = \{a \in L \,|\, \kappa_L(a, v) = 0 \text{ for all } v \in V\}$. This is automatically a sub-vector space.

For the next two points of the remark we assume that k is algebraically closed and of characteristic zero.

3. By Corollary 3.3.5, L is solvable iff $[L, L] \subset L^\perp$.

4. *Claim:* $L^\perp$ is a solvable ideal.

   ⌈ $L^\perp$ is an ideal (Why?). By parts 2 and 3, $\kappa_{L^\perp}(a, b) = 0$ for all $a, b \in L^\perp$. By Corollary 3.3.5, $L^\perp$ is solvable. ⌋ ?

   In particular, $L^\perp \subset \mathrm{Rad}(L)$.

   Thus for semisimple L, the Killing form is non-degenerate. (For info: The converse also holds as we will see in Section 5.)

Preparation for the proof of Cartan's criterion

For the rest of this section, k is assumed to be algebraically closed and of characteristic zero.

Definition 3.3.7. Let V be a finite-dimensional k-vector space and let $x : V \to V$ be linear. Then x is *semisimple* if it is diagonalisable, and it is *nilpotent* if $x^n = 0$ for some n.

- If x is semisimple and nilpotent, then $x = 0$.

Theorem 3.3.8 (Jordan-Chevallay decomposition)**.** Let V a finite-dimensional k-vector space, and $x \in \mathrm{End}(V)$. Then:

1. There exist unique x_s, x_n such that $x = x_s + x_n$, x_s is semisimple, x_n is nilpotent, and $[x_s, x_n] = 0$.

2. There exist polynomials $p, q \in k[T]$ with zero constant term (which depend on x) such that $x_s = p(x)$, $x_n = q(x)$.

Proof. Part 1, existence: Write x in Jordan-normal form. If E is a subspace on which x acts by a Jordan block (in an appropriate basis), choose the corresponding blocks in x_s and x_n as

$$x|_E = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}, \quad x_s|_E = \begin{pmatrix} \lambda & 0 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 0 \\ 0 & & & \lambda \end{pmatrix}, \quad x_n|_E = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}.$$

Part 2: Let $\lambda_1, \dots, \lambda_k$ be the pairwise distinct eigenvalues of x, and let m_i be their algebraic multiplicity. If 0 is not an eigenvalue, we add $\lambda_0 = 0$ to the list and set $m_0 = 1$. The ideals in $k[T]$ generated by $(T - \lambda_i)^{m_i}$ are pairwise coprime. By the Chinese Remainder Theorem for commutative rings, there is a (non-unique) polynomial $p \in k[T]$, such that

$$p(T) \cong \lambda_i \mod (T - \lambda_i)^{m_i} \qquad \text{for all } i$$

In other words, for each i there is a polynomial $r_i(T)$, such that

$$p(T) = \lambda_i + (T - \lambda_i)^{m_i} r_i(T)\ .$$

In particular, $p(0) = 0$ follows by choosing i such that $\lambda_i = 0$.

For $i = 1, \dots, k$, let V_i be the eigenspace of eigenvalue λ_i for x_s (i.e. the generalised eigenspace of that eigenvalue for x). Then

$$p(x)|_{V_i} = \big(\lambda_i + (x - \lambda_i)^{m_i} r_i(x)\big)|_{V_i} = \lambda_i\, \mathrm{id}_{V_i}$$

as $(x - \lambda_i)^{m_i}$ is zero on V_i. Since V is a direct sum of the V_i, this shows $p(x) = x_s$. Since $x_n = x - x_s$ we can take $q(T) = T - p(T)$.

Part 1, uniqueness: Let $x = s + n$ be another decomposition such that the conditions in part 1 hold. Then $[x_s, s] = [p(x), s] = 0$ as $[x, s] = 0$ by assumption. Hence x_s and s can be simultaneously diagonalised, and in particular also $x_s - s$ is semisimple. Analogously, $[x_n, n] = 0$, and hence $x_n - n$ is nilpotent (Why?). But as $x_s + x_n = x = s + n$, also $x_s - s = n - x_n$, ? and so $x_s - s$ is simultaneously semisimple and nilpotent. Hence $x_s - s = 0 = n - x_n$. □

Remark 3.3.9. If $x : V \to V$ preserves a subspace, so do x_s, x_n: Let $A \subset V$ be a sub-vector space such that $xA \subset A$. Then also $x_s A \subset A$ and $x_n A \subset A$. (Why? *Hint:* Use part 2 of Theorem 3.3.8.) ?

Lemma 3.3.10. Let V be a finite-dimensional k-vector space and $x \in \mathrm{End}(V)$. If $x = x_s + x_n$ is the Jordan-Chevallay decomposition of x, then $\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$ is the Jordan-Chevallay decomposition of $\mathrm{ad}(x) : \mathrm{End}(V) \to \mathrm{End}(V)$.

Proof. Since the decomposition is unique, we just need to check the defining properties:

- $\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$: Clear.

- $\mathrm{ad}(x_s)$ is semisimple: Let $\{v_i\}_{i=1,\dots,\dim V}$ be a basis of V such that x_s is diagonal, $xv_i = \lambda_i v_i$ for some $\lambda_i \in k$. Let $E_{ij} \in \mathrm{End}(V)$ be the map $E_{ij}(v_a) = \delta_{a,j} v_i$, cf. Section 1.6. The E_{ij} form a basis of $\mathrm{End}(V)$, and we have, for all a,

$$\begin{aligned} \mathrm{ad}(x_s)E_{ij}v_a &= [x_s, E_{ij}]v_a = x_s E_{ij} v_a - E_{ij} x_s v_a \\ &= \delta_{j,a} x_s v_i - \lambda_a E_{ij} v_a = \lambda_i \delta_{j,a} v_i - \lambda_j \delta_{j,a} v_i \\ &= (\lambda_i - \lambda_j) E_{ij} v_a \ . \end{aligned}$$

  Thus E_{ij} is an eigenvector of $\mathrm{ad}(x_s)$ of eigenvalue $\lambda_i - \lambda_j$. In particular, $\mathrm{ad}(x_s)$ is diagonal in the basis $\{E_{ij}\}$.

- $\mathrm{ad}(x_n)$ is nilpotent: Lemma 3.1.8.

- $[\mathrm{ad}(x_s), \mathrm{ad}(x_n)] = 0$: Set $L = gl(V)$. Since $\mathrm{ad} : L \to gl(L)$ is a Lie algebra homomorphism, and since $[x_s, x_n] = 0$ be assumption, we have (14.11)

$$[\mathrm{ad}(x_s), \mathrm{ad}(x_n)] = \mathrm{ad}([x_s, x_n]) = 0 \ .$$

□

Lemma 3.3.11. Let V be a finite-dimensional k-vector space and let $A \subset B \subset \mathrm{End}(V)$ be subspaces. Set

$$M = \{x \in \mathrm{End}(V) \,|\, \mathrm{ad}(x)(B) \subset A\} \ .$$

If $x \in M$ satisfies $\mathrm{tr}_V(xy) = 0$ for all $y \in M$, then x is nilpotent.

Proof. Let $x = x_s + x_n$ be the Jordan-Chevallay decomposition of x. We will show that $x_s = 0$ (and so $x = x_n$ is nilpotent).

As $\mathrm{char}(k) = 0$, we have an embedding $\mathbb{Q} \hookrightarrow k$, and we will identify $\mathbb{Q}$ with its image in k (e.g. $\mathbb{Q} \subset \mathbb{C}$).

Let $\{v_i\}$ be a basis of V such that x_s is diagonal, $x_s v_i = \lambda_i v_i$ for some $\lambda_i \in k$. Write

$$\Lambda = \mathrm{span}_{\mathbb{Q}}\{\lambda_1, \dots, \lambda_{\dim V}\}$$

for the $\mathbb{Q}$-span of the eigenvalues in k. Then Λ is a $\mathbb{Q}$-vector space (e.g. by construction $\dim_{\mathbb{Q}} \Lambda \leq \dim_k V$).

Let $f : \Lambda \to \mathbb{Q}$ be an arbitrary $\mathbb{Q}$-linear map. Define $y \in \mathrm{End}(V)$ on the basis $\{v_i\}$ as

$$y(v_i) = f(\lambda_i) v_i \ .$$

Claim: $y \in M$.

> ⌈ Let E_{ij} be the basis of $\mathrm{End}(V)$ defined by $E_{ij} v_a = \delta_{j,a} v_i$ as in the proof of Lemma 3.3.10, where we also saw that $\mathrm{ad}(x_s)(E_{ij}) = (\lambda_i - \lambda_j) E_{ij}$. In the same way one checks $\mathrm{ad}(y)(E_{ij}) = (f(\lambda_i) - f(\lambda_j)) E_{ij}$.
>
> Pick a polynomial $r(T) \in k[T]$ such that $r(\lambda_i - \lambda_j) = f(\lambda_i - \lambda_j)$ for all $i, j = 1, \dots, \dim(V)$. Note that for $i = j$ this gives $r(0) = 0$, and that since f is $\mathbb{Q}$-linear we have $r(\lambda_i - \lambda_j) = f(\lambda_i) - f(\lambda_j)$.
>
> By Lemma 3.3.10, $\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$ is the Jordan-Chevallay decomposition of $\mathrm{ad}(x)$. Let $p \in k[T]$ be a polynomial as in Theorem 3.3.8 (2) such that $\mathrm{ad}(x_s) = p(\mathrm{ad}(x))$. Using this, we compute
>
> $$\begin{aligned} \mathrm{ad}(y) E_{ij} &= (f(\lambda_i) - f(\lambda_j)) E_{ij} = r(\lambda_i - \lambda_j) E_{ij} = r(\mathrm{ad}(x)) E_{ij} \ . \\ &= r(p(\mathrm{ad}(x))) E_{ij} \end{aligned}$$
>
> Thus $\mathrm{ad}(y) = r(p(\mathrm{ad}(x)))$. Since $r(p(T))$ by construction has no constant term, $r(p(x)) : \mathrm{End}(V) \to \mathrm{End}(V)$ is a linear map given by a sum of non-zero powers of x. As $\mathrm{ad}(x)(B) \subset A$ by assumption, also $\mathrm{ad}(y)(B) = r(p(\mathrm{ad}(x)))(B) \subset A$. ⌋

Since $y \in M$ by the above claim, by assumption $\mathrm{tr}_V(xy) = 0$. On the other hand, writing $\{v_i^*\}$ for the basis of V^* dual to $\{v_i\}$,

$$\mathrm{tr}_V(xy) = \sum_i v_i^*\big(xy(v_i)\big) = \sum_i f(\lambda_i) v_i^*\big(x(v_i)\big) = \sum_i f(\lambda_i)\lambda_i \quad \in \Lambda \ .$$

Since $\mathrm{tr}_V(xy) \in \Lambda$, we can apply f to it, and since f is $\mathbb{Q}$-linear, this gives $0 = f(\mathrm{tr}_V(xy)) = \sum_i f(\lambda_i)^2$. But the $f(\lambda_i)$ are rational, and so the sum of squares can be zero only if $f(\lambda_i) = 0$ for all i. Hence f vanishes on a spanning set of Λ, and so must be zero.

The only way in which every $\mathbb{Q}$-linear map $\Lambda \to \mathbb{Q}$ can be zero is if $\Lambda = \{0\}$, i.e. if $\lambda_i = 0$ for all i. But this is equivalent to $x_s = 0$. □

Proof of Cartan's criterion

Recall Cartan's criterion (Theorem 3.3.1):

$$\big(\dim V < \infty,\ L \subset gl(V),\ \forall x \in [L,L], y \in L : \mathrm{tr}_V(xy) = 0\big) \ \Rightarrow\ L \text{ solvable}$$

Proof of Theorem 3.3.1. Set $A = [L,L]$ and $B = L$ such that $A \subset B \subset \operatorname{End} V$, and let M be as in Lemma 3.3.11:

$$M = \{x \in \operatorname{End}(V) \,|\, \operatorname{ad}(x)(B) \subset A\} \ .$$

By construction, $L \subset M$.

Claim 1: Let $x \in [L,L]$. Then $\mathrm{tr}_V(xy) = 0$ for all $y \in M$.

> ⌈ $[L,L]$ is spanned by elements of the form $[a,b]$ with $a,b \in L$. Let $y \in M$ be given. Then
>
> $$\mathrm{tr}_V([a,b]y) = \mathrm{tr}_V(a[b,y]) = 0 \ ,$$
>
> where the last equality follows by the assumptions in the statement of the theorem as $a \in L$ and $[b,y] = -\operatorname{ad}(y)b \in A = [L,L]$ by definition of M. ⌋

Claim 2: $[L,L]$ is nilpotent.

> ⌈ By Claim 1 and Lemma 3.3.11, each $x \in [L,L]$ is nilpotent. By Lemma 3.1.8, also $\operatorname{ad}(x)$ is nilpotent. By Engel's Theorem (Theorem 3.1.3), $[L,L]$ is nilpotent. ⌋

Claim 2 implies that L is solvable. Indeed, $D^1L = [L,L] =: \widetilde{L}$, and so $D^{i+1}L = D^i\widetilde{L} \subset C^i\widetilde{L}$, and the latter eventually becomes zero as $\widetilde{L}$ is nilpotent. □

4 The universal enveloping algebra

In this section, k denotes an arbitrary field.

4.1 The tensor algebra

Definition 4.1.1. Let V be a k-vector space. A *tensor algebra of V* is a pair (T,t), where T is a unital associative algebra over k and $t : V \to T$ is a linear map, such that the following universal property holds:

For each unital associative algebra A and each linear map $f : V \to A$ there exists a unique unital algebra homomorphism $\hat{f} : T \to A$ such that $f = \hat{f} \circ t$.

As a commuting diagram, the condition can be stated as:

$$\begin{tikzcd} V \arrow[r, "t"] \arrow[d, "\forall f"'] & T \arrow[dl, dashed, "\exists! \hat{f}"] \\ A & \end{tikzcd}$$

Lemma 4.1.2. Let (T, t) and (T', t') be tensor algebras for V. Then there exists a unique unital algebra isomorphism $\phi : T \to T'$ such that

$$\begin{tikzcd} & V \arrow[dl, "t"'] \arrow[dr, "t'"] & \\ T \arrow[rr, "\phi"] & & T' \end{tikzcd}$$

commutes.

Proof. From the universal property we get unique unital algebra homomorphisms ϕ, ψ such that the diagrams

$$\begin{tikzcd} V \arrow[r, "t"] \arrow[d, "t'"'] & T \arrow[dl, dashed, "\phi"] \\ T' & \end{tikzcd} \quad , \quad \begin{tikzcd} V \arrow[r, "t'"] \arrow[d, "t"'] & T' \arrow[dl, dashed, "\psi"] \\ T & \end{tikzcd}$$

commute. We will show that these are inverse to each other. The composition $\psi \circ \phi$ makes the following diagram commute: (Why?)

$$\begin{tikzcd} V \arrow[r, "t"] \arrow[d, "t"'] & T \arrow[dl, dashed, "\psi\circ\phi"] \\ T & \end{tikzcd} .$$

But that diagram also commutes when we label the dashed arrow by the algebra homomorphism id_T. But uniqueness we must have $\psi \circ \phi = \mathrm{id}_T$. In the same way one shows that $\phi \circ \psi = \mathrm{id}_{T'}$. $\square$

■ Let V be a k-vector space. Set

$$T^0V := k \ , \quad T^1V := V \ , \quad T^2V := V \otimes V \ , \ldots , \quad T^mV := \underbrace{V \otimes \cdots \otimes V}_{m \text{ times}} \ , \ldots$$

and

$$T(V) := \bigoplus_{m=0}^{\infty} T^m V \ .$$

We have an embedding $\tau : V \to T(V)$ given by $v \mapsto v \in T^1 V$. We turn $T(V)$ into a unital associative algebra by defining the unit to be $1 \in k = T^0 V \subset T(V)$, and the product to be, on homogeneous components,

$$\begin{array}{ccc} T^m V \times T^n V & \longrightarrow & T^{m+n} V \\ (u_1 \otimes \cdots \otimes u_m \, , \, v_1 \otimes \cdots \otimes v_n) & \longmapsto & u_1 \otimes \cdots \otimes u_m \otimes v_1 \otimes \cdots \otimes v_n \ . \end{array}$$

Proposition 4.1.3. Let V be a k-vector space. Then $(T(V), \tau)$ is a tensor algebra for V.

Proof. Let A be a unital associative algebra and $f : V \to A$ a linear map. We must show the existence and uniqueness of a unital algebra homomorphism $\hat{f} : T(V) \to A$ such that $f = \hat{f} \circ \tau$.

Existence: For $1 \in T^0 V$ we set $\hat{f}(1) = 1_A$. For $x := v_1 \otimes \cdots \otimes v_m \in T^m V$ define $\hat{f}(x) = f(v_1) \cdots f(v_m) \in A$. (Why is this a well-defined map out of the tensor product? Why does it define an algebra homomorphism? Why does it satisfy $f = \hat{f} \circ \tau$?)

Uniqueness: Let $g : T(V) \to A$ be a unital algebra homomorphism which satisfies $f = g \circ \tau$. Then

$$\begin{aligned} g(v_1 \otimes \cdots \otimes v_m) &\overset{(1)}{=} g(v_1) \cdots g(v_m) \overset{(2)}{=} g(\tau(v_1)) \cdots g(\tau(v_m)) \\ &\overset{(3)}{=} f(v_1) \cdots f(v_m) \overset{(4)}{=} \tilde{f}(v_1 \otimes \cdots \otimes v_m) \ , \end{aligned}$$

where the steps are: (1) g is an algebra homomorphism, (2) definition of τ, (3) g satisfies $g \circ \tau = f$, (4) definition of $\tilde{f}$. As this holds for all m, and the $v_1 \otimes \cdots \otimes v_m$ span $T^m V$, this shows $g = \tilde{f}$. □

Remark 4.1.4.

1. Below we will say "the tensor algebra" to refer to $T(V)$ as constructed above.

2. If $V = \{0\}$, then $T(V) = T^0(V) = k$ is one-dimensional. If $V \neq \{0\}$, then $T(V)$ is infinite-dimensional.

3. If $V = k$, then $T(V) \cong k[X]$, the polynomial algebra in one variable (Why?). But for $V = k^2$, $T(V) \not\cong k[X, Y]$. Indeed, let $a = (1, 0)$ and $b = (0, 1)$ denote the standard basis of k^2. Then $ab \neq ba$ in $T(V)$, but $XY = YX$ in $k[X, Y]$. (Can you write a basis for $T(V)$ in terms of a, b?)

4. For info: Let $F : \mathrm{Alg}_k \to \mathrm{Vec}_k$ be the forgetful functor from algebras and algebra homomorphism to vector spaces and linear maps. Taking the tensor algebra defines a functor $T : \mathrm{Vec}_k \to \mathrm{Alg}_k$ which in fact is left adjoint to the forgetful functor, i.e. there is a natural family of isomorphisms (obtained from the universal property)

$$\phi_{V,A} : \mathrm{Hom}_{\mathrm{Vec}}(V, FA) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Alg}}(TV, A) \ .$$

Functors left adjoint to a forgetful functor are often called "free". In this sense, $T(V)$ is the algebra freely generated by V.

4.2 The universal enveloping algebra

■ If A is an associative algebra, we will always also view it as a Lie algebra with commutator as Lie bracket, $[a, b]_A := ab - ba$. In particular, if L is a Lie algebra and A is an associative algebra, a Lie algebra homomorphism $\varphi : L \to A$ satisfies $\varphi([[x, y]_L) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$.

Definition 4.2.1. Let L be a Lie algebra over k. A *universal enveloping algebra of* L is a pair (U, u), where U is a unital associative algebra over k and $u : L \to U$ is a Lie algebra homomorphism, such that the following universal property holds:

> For each unital associative algebra A and each Lie algebra homomorphism $\varphi : L \to A$ there exists a unique unital algebra homomorphism $\hat{\varphi} : T \to A$ such that $\varphi = \hat{\varphi} \circ u$.

As a commuting diagram, the condition can be stated as:

$$\begin{tikzcd} L \arrow[r, "u"] \arrow[d, "\forall\varphi"'] & U \arrow[dl, dashed, "\exists!\hat{\varphi}"] \\ A & \end{tikzcd}$$

Lemma 4.2.2. Let (U, u) and (U', u') be universal enveloping algebras for a Lie algebra L. Then there exists a unique unital algebra isomorphism $\varphi : U \to U'$ such that

$$\begin{tikzcd} & L \arrow[dl, "u"'] \arrow[dr, "u'"] & \\ U \arrow[rr, "\varphi"] & & U' \end{tikzcd}$$

commutes.

(The proof is an exercise.) E (Ex. 17)

■ From here on "algebra" means "unital associative algebra".

Remark 4.2.3. Let A be an algebra over k. Recall some properties of algebras and ideals:

1. A *two-sided ideal of* A (or *ideal* for short) is a sub-vector space $I \subset A$ such that
$$\forall a \in A, i \in I \;:\; a \cdot i \in I \text{ and } i \cdot a \in I \;.$$

2. For $S \subset A$ a subset, write $\langle S \rangle \subset A$ for the two-sided ideal generated by S:
$$\langle S \rangle = \mathrm{span}_k\{a \cdot s \cdot b \,|\, s \in S, a, b \in A\} \;.$$
Since A is unital, we have $S \subset \langle S \rangle$. If $f : A \to B$ is an algebra homomorphism, such that $f(s) = 0$ for all $s \in S$, then also $f(\langle S \rangle) = \{0\}$.
(Why is $\langle S \rangle$ the smallest ideal containing S? Why does the homomorphism-property hold?) ?

3. Let $I \subset A$ be a two-sided ideal, and $\pi : A \to A/I$ is the canonical projection to the quotient vector space. There is a unique algebra structure on A/I such that π is an algebra homomorphism. For example, the product on A/I is given by $(a+I)\cdot(b+I) := ab + I$.

■ Let L be a Lie algebra over k. In $T(L)$ consider the subset

$$S = \{x \otimes y - y \otimes x - [x, y]_L \,|\, x, y \in L\} \;.$$

Note that $x \otimes y - y \otimes x \in T^2 L$ and $[x, y]_L \in T^1 L$. Define

$$U(L) := T(L)/\langle S \rangle \quad , \quad \iota_L := \big[L \xrightarrow{\tau} T(L) \xrightarrow{\pi} U(L)\big] \;.$$

Then ι_L is a Lie algebra homomorphism, i.e. (17.11)

$$\iota_L([x,y]_L) \;=\; \iota_L(x)\iota_L(y) - \iota_L(y)\iota_L(x) \qquad \text{for all } \; x, y \in L \;.$$

(Why? And why is τ alone not a Lie algebra homomorphism if $\dim L > 1$?) ?
Since $\pi : T(L) \to U(L)$ is surjective, $U(L)$ is spanned (as a k-vector space) by elements of the form

$$\iota_L(x_1)\iota_L(x_2)\cdots\iota_L(x_m) \qquad , \quad m \in \mathbb{Z}_{\geq 0} \;, \quad x_1, \dots, x_m \in L \;.$$

Here, the product is that of $U(L)$.

Proposition 4.2.4. Let L be a Lie algebra over k. Then $(U(L), \iota_L)$ is a universal enveloping algebra for L.

Proof. Let A be an algebra and $\varphi : L \to A$ a Lie algebra homomorphism. We have to show existence and uniqueness of an algebra homomorphism $\hat{\varphi} : U(L) \to A$ such that $\varphi = \hat{\varphi} \circ \iota_L$.

Existence: From the universal property of the tensor algebra we get an algebra homomorphism $\phi : T(L) \to A$ such that

$$\begin{tikzcd} L \arrow[r, "\tau"] \arrow[d, "\varphi"'] & T(L) \arrow[dl, dashed, "\phi"] \\ A & \end{tikzcd}$$

commutes. Let $x, y \in L$ and $s = x \otimes y - y \otimes x - [x,y]_L$. Then

$$\begin{aligned} \phi(s) &\overset{(1)}{=} \phi(x)\phi(y) - \phi(y)\phi(x) - \phi([x,y]_L) \\ &\overset{(2)}{=} \phi(\tau(x))\phi(\tau(x)) - \phi(\tau(x))\phi(\tau(x)) - \phi(\tau([x,y]_L)) \\ &\overset{(3)}{=} \varphi(x)\varphi(y) - \varphi(y)\varphi(x) - \varphi([x,y]_L) \overset{(4)}{=} 0 \;, \end{aligned}$$

where the individual steps follow from: (1) ϕ is an algebra homomorphism, (2) definition of τ, (3) $\phi \circ \tau = \varphi$, and (4) φ is a Lie algebra homomorphism.

Thus $\phi(S) = \{0\}$ and by Remark 4.2.3 (2) also $\phi(\langle S \rangle) = \{0\}$. Hence there exists a unique linear map $\hat{\varphi} : T(L)/\langle S \rangle \to A$ such that

$$\begin{tikzcd} T(L) \arrow[r, "\pi"] \arrow[d, "\varphi"'] & T(L)/\langle S \rangle = U(L) \arrow[dl, dashed, "\hat{\varphi}"] \\ A & \end{tikzcd}$$

commutes. Since φ is an algebra homomorphism, so is $\hat{\varphi}$ (Why?). By construction, $\varphi = \hat{\varphi} \circ \iota_L$. ?

Uniqueness: Let $\psi : U(L) \to A$ be an algebra homomorphism which satisfies $\varphi = \psi \circ \iota_L$. Recall from above that $U(L)$ is spanned by elements of the form $u = \iota_L(x_1)\iota_L(x_2) \cdots \iota_L(x_m)$. We have

$$\begin{aligned} \psi(u) &\overset{(1)}{=} \psi(\iota_L(x_1)) \cdots \psi(\iota_L(x_m)) \\ &\overset{(2)}{=} \varphi(x_1) \cdots \varphi(x_m) \\ &\overset{(3)}{=} \hat{\varphi}(\iota_L(x_1)) \cdots \hat{\varphi}(\iota_L(x_m)) \overset{(4)}{=} \hat{\varphi}(u) \end{aligned}$$

where the steps are: (1) ψ is an algebra homomorphism, (2) ψ satisfies $\psi \circ \iota_L = \varphi$, (3) $\hat{\varphi}$ satisfies $\hat{\varphi} \circ \iota_L = \varphi$, and (4) $\hat{\varphi}$ is an algebra homomorphism.

Since ψ and $\hat{\varphi}$ agree on a spanning set for $U(L)$, they are equal. □

Remark 4.2.5.

1. When we say "the universal enveloping algebra of L" below, we refer to the pair $(U(L), \iota_L)$ constructed above.

2. By the universal property, to see if two algebra homomorphism $f, g : U(L) \to L$ are equal, it is enough to compare them on the image of ι_L:

$$f = g \quad \Leftrightarrow \quad f \circ \iota_L = g \circ \iota_L \ .$$

3. Let $\phi : L \to L'$ be a Lie algebra homomorphism. Then there exists a unique algebra homomorphism $U(\phi) : U(L) \to U(L')$ such that

$$\begin{tikzcd} L \arrow[r, "\phi"] \arrow[d, "\iota_L"'] & L' \arrow[d, "\iota_{L'}"] \\ U(L) \arrow[r, dashed, "U(\phi)"] & U(L') \end{tikzcd}$$

commutes. (Why?) ?

4. For info: One can check that part 3 turns U into a functor $U : \mathrm{Lie}_k \to \mathrm{Alg}_k$. The functor U is left adjoint to the functor $\mathrm{Alg}_k \to \mathrm{Lie}_k$ which replaces the associative product by the commutator Lie bracket.

5. $T(L)$ and $U(L)$ are Hopf algebras (for $T(L)$ one only needs L to be a vector space). For example, the coproducts $\Delta_T : T(V) \to T(V) \otimes T(V)$ and $\Delta_U : U(L) \to U(L) \otimes U(L)$ are defined via the universal property from $a \mapsto a \otimes 1 + 1 \otimes a$ $(a \in L)$.

4.3 The symmetric algebra

Definition 4.3.1. Let V be a k-vector space. The *symmetric algebra for V* is

$$S(V) := T(V)/J \quad , \quad \text{where} \quad J = \langle x \otimes y - y \otimes x \,|\, x, y \in V \rangle \ .$$

(One can also characterise $S(V)$ by a universal property. Can you think of one? *Hint:* Part 1 of the remark below.) ?

Remark 4.3.2.

1. $S(V)$ is a commutative algebra (Why?). If we turn V into a Lie algebra by equipping it with the zero-bracket $[x, y] = 0$ for all x, y, then $S(V) = U(V)$. ?

2. Define $J^m \subset T^m V$ to be

$$J^m = \mathrm{span}_k\{a \otimes x \otimes y \otimes b - a \otimes y \otimes x \otimes b \,|\, x, y \in V, a \in T^i V, b \in T^j V \text{ for } i + j + 2 = m\} \ .$$

Then $J = \bigoplus_{m=2}^{\infty} J^m$ (Why?) and hence (setting $J^0 = J^1 = \{0\}$) ?

$$S(V) = \bigoplus_{m=0}^{\infty} S^m V \quad , \quad \text{where} \quad S^m V := T^m V / J^m \ .$$

In particular, $S(V)$ is still graded as an algebra: for $a \in S^m V$, $b \in S^n V$ have $a \cdot b \in S^{m+n} V$.

(Why is the universal enveloping algebra $U(L)$ of a non-abelian Lie algebra L not graded in this way?) ?

■ Fix a basis $\{v_\lambda\}_{\lambda \in \Omega}$ of V, and pick a total order on Ω (this is possible even for infinite Ω by the axiom of choice). E.g. $\Omega = \{1, 2, \dots, n\}$ or $\Omega = \mathbb{Z}$ or $\Omega = \mathbb{R}$. Let $S\Omega$ be the set of finite increasing sequences in Ω,

$$S\Omega = \{(\lambda_1, \dots, \lambda_m) \,|\, m \geq 0, \lambda_1 \leq \dots \leq \lambda_m\} \ .$$

If $\pi : T(V) \to S(V)$ is the canonical projection, set $z_\lambda := \pi(v_\lambda)$ and for a finite sequence $M = (\lambda_1, \dots, \lambda_m)$ (ordered or not) write

$$v_M := v_{\lambda_1} \otimes \dots \otimes v_{\lambda_m} \in T(V) \quad , \quad z_M := z_{\lambda_1} \cdots z_{\lambda_m} = \pi(v_M) \in S(V) \ .$$

Proposition 4.3.3. $\{z_M\}_{M \in S\Omega}$ is a basis of $S(V)$.

Proof. First note that the z_M span $S(V)$ (Why?). It remains to show that they are linearly independent. ?

Denote the length of a sequence $M = (\lambda_1, \dots, \lambda_m)$ by $|M| = m$. For a permutation $\sigma \in \Pi_m$ of m elements write $M\sigma = (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(m)})$ (think of M as a function $\{1, \dots, m\} \to \Omega$ and of the notation $M\sigma$ as precomposition $M \circ \sigma$). Then $\{v_{M\sigma} | M \in S\Omega, \sigma \in \Pi_{|M|}\}$ is a basis of $T(V)$. (Why?) ?

Set

$$J' = \mathrm{span}_k\{v_M - v_{M\sigma} \,|\, M \in S\Omega, \sigma \in \Pi_{|M|}\} \ .$$

Then

- J' is a two-sided ideal in $T(V)$. (Why? *Hint:* Show first that for any $\sigma, \pi \in \Pi_{|M|}$ we have $v_{M\pi} - v_{M\sigma} \in J'$.) ?

- $J \subset J'$: We have $x \otimes y - y \otimes x \in J'$ for all $x, y \in V$, as follows when expanding x, y in the basis v_λ.

- $J' \subset J$: If τ is a transposition of neighbouring elements, by definition $v_M - v_{M\tau} \in J$. Write an arbitrary $\sigma \in \Pi_{|M|}$ as a composition $\sigma = \tau_1 \cdots \tau_L$ of such transpositions. Then

$$v_M - v_{M\sigma} = \underbrace{v_M - v_{M\tau_1}}_{\in J} + \underbrace{v_{M\tau_1} - v_{M\tau_1\tau_2}}_{\in J} + \cdots \in J \ .$$

Suppose $\sum_{M \in F} c_M z_M = 0$ for some $c_M \in k$ (only finitely many non-zero). This is equivalent to

$$\begin{aligned}
\sum_{M\in\Omega} c_M z_M = 0 &\Leftrightarrow \sum_{M\in\Omega} c_M v_M \in J \\
&\Leftrightarrow \sum_{M\in\Omega} c_M v_M = \sum_{M\in\Omega, \sigma\neq \mathrm{id}} a_{M,\sigma}(v_M - v_{M\sigma}) \ \text{ for some } a_{M,\sigma} \in k \\
&\Leftrightarrow \sum_{M\in\Omega} \Big(c_M - \sum_{\sigma\neq\mathrm{id}} a_{M,\sigma}\Big) v_M + \sum_{M\in\Omega,\sigma\neq\mathrm{id}} a_{M,\sigma} v_{M\sigma} = 0 \ .
\end{aligned}$$

But the $v_{M\sigma}$ are a basis of $T(V)$, and so all $a_{M,\sigma}$ are zero, and hence also all c_M. □

Remark 4.3.4. The basis $\{z_M\}_{M\in S\Omega}$ gives an algebra isomorphism between $S(V)$ and the polynomial algebra $k[(z_\lambda)_{\lambda\in\Omega}]$ in the variables z_λ, $\lambda \in \Omega$.

4.4 The Poincaré-Birkhoff-Witt Theorem

■ Let L be a Lie algebra over k and $(U(L), \iota)$ its universal enveloping algebra, $\pi : T(L) \to U(L)$ the canonical projection.

Fix a basis $\{v_\lambda\}_{\lambda\in\Omega}$ of L with Ω a totally ordered set, and write $S\Omega$ for finite increasing sequences as in the previous section. For a finite sequence $M = (\lambda_1, \ldots, \lambda_m)$ (ordered or not) write

$$v_M := v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_m} \in T(L) \quad , \quad x_M := \iota(x_{\lambda_1}) \cdots \iota(x_{\lambda_m}) = \pi(v_M) \in U(L) \ .$$

For the empty sequence $M = ()$ we declare $v_M = 1$, $x_M = 1$.

Theorem 4.4.1 (Poincaré-Birkhoff-Witt Theorem)**.**
$\{x_M\}_{M\in S\Omega}$ is a basis of $U(L)$.

The proof requires some preparation.

■ Set

$$U_m L \ := \ \mathrm{span}_k\{\iota(x_1)\cdots\iota(x_i) \,|\, i \leq m, x_1, \ldots, x_i \in L\} \ = \ \pi\Big(\bigoplus_{i=0}^{m} T^i V\Big) \ .$$

The ascending chain of subspace

$$U_0L \subset U_1L \subset U_2L \subset \cdots$$

is called the *canonical filtration of* $U(L)$.

Remark 4.4.2.

1. By construction, we have $\bigcup_{m=0}^{\infty} U_mL = U(L)$, and for each $v \in U(L)$ there is an m such that $v \in U_mL$.
2. For $a \in U_mL$, $b \in U_nL$ we have $a \cdot b \in U_{m+n}L$. (Why?) ?

(21.11)

Lemma 4.4.3. For all $m > 0$, $\sigma \in \Pi_m$ and $x_1, \ldots, x_m \in L$ we have

$$\iota(x_1)\cdots\iota(x_m) \;-\; \iota(x_{\sigma(1)})\cdots\iota(x_{\sigma(m)}) \;\in\; U_{m-1}L\ .$$

Proof. It is enough to show the statement for transpositions of neighbouring elements (Why?). Let $\tau \in \Pi_M$ be the transposition $\tau(j) = j+1$, $\tau(j+1) = j$. ? In this case, the statement becomes

$$\begin{aligned}
&\iota(x_1)\cdots\iota(x_{j-1})\big(\iota(x_j)\iota(x_{j+1}) - \iota(x_{j+1})\iota(x_j)\big)\iota(x_{j+2})\cdots\iota(x_m) \\
&= \;\iota(x_1)\cdots\iota(x_{j-1})\,\big[\iota(x_j),\iota(x_{j+1})\big]\,\iota(x_{j+2})\cdots\iota(x_m) \;\in\; U_{m-1}L\ .
\end{aligned}$$

□

Lemma 4.4.4. We have $U(L) = \mathrm{span}_k\{x_M \,|\, M \in S\Omega\}$.

Proof. We will show by induction that U_mL is spanned by the x_M with $M \in S\Omega$ and $|M| \le m$.

$m = 0, 1$: clear.

$m-1 \rightsquigarrow m$: By definition, U_mL is spanned by $x_{M\sigma}$ with $M \in S\Omega$, $\sigma \in \Pi_{|M|}$ and $|M| \le m$. By Lemma 4.4.3, the difference $x_M - x_{M\sigma}$ lies in $U_{m-1}L$. By induction hypothesis, $U_{m-1}L$ is spanned by x_N with $N \in S\Omega$, $|N| \le m-1$. Altogether, U_mL is spanned by x_M, $M \in S\Omega$, $|M| \le m$ as claimed. □

Main technical lemma: L acts on SL

■ The idea of the construction is: "An element of an ordered basis of L acts on an ordered product in SL by moving it to its correct position using the Lie bracket and iterate until only ordered products occur."

Consider $L = sl(2,k)$ as an example, with basis $\{f, h, e\}$ ordered as $f < h < e$. By Proposition 4.3.3, the corresponding basis of $S(L)$ is

$$\{1, \underbrace{f, h, e}_{\in S^1L}, \underbrace{ff, fh, fe, hh, he, ee}_{\in S^2L}, \underbrace{fff, ffh, ffe, fhh, \ldots}_{\in S^3L}, \ldots\}$$

For example, to compute the action of e on ff, we first note that in $U(L)$ we have the following equalities:

$$\begin{aligned} eff &= [e,f]f + f[e,f] + ffe = hf + fh + ffe = [h,f] + 2fh + ffe \\ &= -2f + 2fh + ffe \ . \end{aligned}$$

We have re-expressed eff in terms of ordered products, and so we would like to set

$$e.(ff) = -2f + 2fh + ffe \ .$$

The next lemma shows that this indeed gives a well defined action.

■ The grading on $S(L) = \bigoplus_{m=0}^{\infty} S^m L$ defines a filtration by setting

$$S_m := \bigoplus_{i=0}^{m} S^i L \ .$$

For $\mu \in \Omega$ and $M \in S\Omega$, $M = (\lambda_1, \ldots, \lambda_m)$ with $m > 0$ define

$$\mu \leq M \overset{def}{\Leftrightarrow} \mu \leq \lambda_1 \ , \quad \text{and} \quad \mu \leq () \text{ for all } \mu \ .$$

Lemma 4.4.5. There is a family $(f_m)_{m \in \mathbb{Z}_{\geq 0}}$ of linear maps $f_m : L \otimes S_m \to S(L)$ such that for all $m \geq 0$ (set $f_{-1} = 0$, $S_{-1} = \{0\}$):

1. For all $u \in L \otimes S_{m-1}$: $f_m(u) = f_{m-1}(u)$
2. $f_m(L \otimes S_m) \subset S_{m+1}$.
3. For all $z_M \in S_m$ and $\lambda \leq M$: $f_m(v_\lambda \otimes z_M) = z_\lambda z_M$.
4. For all $z_M \in S_m$, $\lambda \in \Omega$: $f_m(v_\lambda \otimes z_M) - z_\lambda z_M \in S_m$.
5. For all $z_T \in S_{m-1}$, $\lambda, \mu \in \Omega$:

   $$f_m(v_\lambda \otimes f_{m-1}(v_\mu \otimes z_T)) - f_m(v_\mu \otimes f_{m-1}(v_\lambda \otimes z_T)) = f_{m-1}([v_\lambda, v_\mu]_L \otimes z_T)$$

Proof. We construct the f_m by induction.

$m = 0$: One checks that $f_0 : L \otimes S_0 \to S(L)$, $x \otimes 1 \mapsto x \in S^1 L$ satisfies 1.–5.

$m - 1 \rightsquigarrow m$: For $M \in S\Omega$, $|M| \leq m$ and $\lambda \in \Omega$, define $f_m(v_\lambda \otimes z_M)$ as follows.

- $|M| < m$: $f_m(v_\lambda \otimes z_M) := f_{m-1}(v_\lambda \otimes z_M)$ (and so f_m satisfies 1.)

- $|M| = m$: If $\lambda \le M$, we set $f_m(v_\lambda \otimes z_M) := z_\lambda z_M$ (and so f_m satisfies 3.)

  If it is not true that $\lambda \le M$, write $M = (\mu, T)$. Then $\lambda > \mu$ and $T \in S\Omega$, $|T| = m-1$, $\mu \le T$. Set

  $$y := f_{m-1}(v_\lambda \otimes z_T) - z_\lambda z_T \ \in \ S_{m-1} \ ,$$

  by property 4 of f_{m-1} (induction hypothesis). Using this, define

  $$f_m(v_\lambda \otimes z_M) := z_\lambda z_M + \underbrace{f_{m-1}(v_\mu \otimes y) + f_{m-1}([v_\lambda, v_\mu]_L \otimes z_T)}_{\in S_m \text{ by property 2 for } f_{m-1}} \qquad (*)$$

It is easy to check that the so-defined f_m satisfies 1.–4. (Details?) Verifying property 5 will take more effort. ?

Let $\lambda, \mu \in \Omega$ and $T \in S\Omega$, $|T| = m-1$ (the case $|T| < m-1$ is covered by the induction assumption). We distinguish several cases.

■ *Case 1: $\lambda = \mu$:* Trivial.

■ *Case 2: $\lambda \le T$ or $\mu \le T$:* We can exchange the role of λ and μ by multiplying property 5 by -1. Hence w.l.o.g. (without loss of generality) we may assume $\mu \le T$. Set $M = (\mu, T)$ and $y = f_{m-1}(v_\lambda \otimes z_T) - z_\lambda z_T$ as in the statement of the lemma. Since $\mu \le T$ we have $z_M = f_{m-1}(v_\mu \otimes z_T)$, and so $(*)$ can be rewritten as

$$\begin{aligned} &f_m(v_\lambda \otimes f_{m-1}(v_\mu \otimes z_T)) - (\lambda \leftrightarrow \mu) \\ &= f_{m-1}([v_\lambda, v_\mu]_L \otimes z_T) + z_\lambda z_\mu z_T - f_m(v_\mu \otimes z_\lambda z_T) \ . \end{aligned}$$

(Why did some of the f_{m-1} have to be changed to f_m in this expression?) ?

- Suppose $\lambda > \mu$: Then $f_m(v_\mu \otimes z_\lambda z_T) = z_\mu z_\lambda z_T$ (Why? *Hint:* How does μ compare to the ordered list obtained from λ and T?) and the above equation simplifies to property 5. ?

- Suppose $\lambda < \mu$: Then also $\lambda \le T$ and we can apply the above proof of case 2 for $\lambda > \mu$ with μ and λ exchanged.

■ *Case 3: Neither $\lambda \le T$ nor $\mu \le T$:* Write $T = (\nu, R)$ with $\nu \le R$ and $|R| = m-2$. By assumption, $\lambda, \mu > \nu$. By property 3 and the induction assumption, we have

$$z_T = z_\nu z_R = f_{m-2}(z_\nu \otimes z_R)$$

We need to show

$$\begin{aligned}&f_m(v_\lambda \otimes f_{m-1}(v_\mu \otimes f_{m-2}(v_\nu \otimes z_R))) - (\lambda \leftrightarrow \mu)\\ &= f_{m-1}([v_\lambda, v_\mu]_L \otimes f_{m-2}(v_\nu \otimes z_R))) \ . \end{aligned} \tag{$**$}$$

To make the expressions more readable, we abbreviate, for $\alpha, \beta, \gamma \in \Omega$

$$\begin{aligned}(\alpha) &= f_{m-2}(v_\alpha \otimes z_R) \ ,\\ (\alpha\beta) &= f_{m-1}(v_\alpha \otimes f_{m-2}(v_\beta \otimes z_R)) \ ,\\ (\alpha\beta\gamma) &= f_m(v_\alpha \otimes f_{m-1}(v_\beta \otimes f_{m-2}(v_\gamma \otimes z_R))) \ ,\end{aligned}$$

and we write $[\alpha, \beta]$ for $[v_\alpha, v_\beta]$, so that e.g. $([\alpha, \beta]) = f_{m-2}([v_\alpha, v_\beta] \otimes z_R)$. In this notation, $(**)$ becomes

$$(\lambda\mu\nu) - (\mu\lambda\nu) = ([\lambda, \mu]\nu) \ .$$

Using property 5 for f_{m-1}, we can rewrite the left hand side as

$$\text{lhs} = (\lambda\nu\mu) + (\lambda[\mu, \nu]) - (\mu\nu\lambda) - (\mu[\lambda, \nu]) \ .$$

Claim: Property 5 holds for $(\lambda\nu\mu)$.

(Exercise. *Hint:* Argue that $f_{m-2}(v_\mu \otimes z_R) = z_\mu z_R + w$ with $w \in S_{m-2}$. Make use of case 2 and the induction hypothesis.) E (Ex. 21)

Exchanging $\lambda \leftrightarrow \mu$ shows that property 5 also holds for $(\mu\nu\lambda)$. This gives

$$\begin{aligned}\text{lhs} &= (\nu\lambda\mu) + ([\lambda, \nu]\mu) + (\lambda[\mu, \nu]) - (\nu\mu\lambda) - ([\mu, \nu]\lambda) - (\mu[\lambda, \nu])\\ &= (\nu[\lambda, \mu]) + ([[\lambda, \nu], \mu]) + ([\lambda, [\mu, \nu]])\\ &= ([\lambda, \mu]\nu) + \underbrace{([\nu, [\lambda, \mu]]) + ([[\lambda, \nu], \mu]) + ([\lambda, [\mu, \nu]])}_{=0 \text{ by Jacobi identity}} \ .\end{aligned}$$

In each equality, we apply property 5 for f_{m-1}, which holds by induction hypothesis. Altogether, this shows case 3 of property 5 for f_m. □

■ Define the map $L \times S(L) \to S(L)$, $(x, v) \mapsto x.v := f_m(x \otimes v)$ for any m such that $v \in S_m$ (this is independent of the choice of m by property 1 in Lemma 4.4.5).

Corollary 4.4.6. The above defines the structure of an L-module on $S(L)$.

Corollary 4.4.7. There exists a unique structure of a $U(L)$-module on $S(L)$ such that $x_M.1 = z_M$.

(The proof of both corollaries is an exercise.) E (Ex. 21)

Proof of Poincaré-Birkhoff-Witt Theorem

Proof of Theorem 4.4.1. By Lemma 4.4.4, the set $\{x_M \,|\, M \in S\Omega\}$ spans $U(L)$. Linear independence follows from Corollary 4.4.7. Namely, suppose $\sum_{M\in S\Omega} c_M x_M = 0$ in $U(L)$ for some $c_M \in k$ (only finitely many non-zero). Then the action on $1 \in S(L)$ gives

$$0 = \Big(\sum_{M\in S\Omega} c_M x_M\Big).1 = \sum_{M\in S\Omega} c_M x_M.1 = \sum_{M\in S\Omega} c_M z_M$$

But the z_M are linearly independent (Proposition 4.3.3), and so $c_M = 0$ for all M. □

Corollary 4.4.8 (to PBW Theorem)**.** Let L be a Lie algebra over k. The Lie algebra homomorphism $\iota_L : L \to U(L)$ is injective.

Proof. In terms of a basis $\{v_\lambda\}_{\lambda\in\Omega}$ as above we have $\iota_L(v_\lambda) = x_{(\lambda)}$, and by the PBW Theorem, all x_M, $M \in S\Omega$ are linearly independent. Thus ι_L maps a basis to a linearly independent set and is hence injective. □

■ Recall from Remark 4.2.5 (3) that a Lie algebra homomorphism $f : L \to L'$ gives rise to an algebra homomorphism $U(f) : U(L) \to U(L')$.

Corollary 4.4.9 (to PBW Theorem)**.** If $f : L \to L'$ is injective / surjective / bijective, then $U(f) : U(L) \to U(L')$ is injective / surjective / bijective.

Proof. Surjective: Follows from the spanning set given above Proposition 4.2.4 (How?). ?

Injective: Pick a basis $\{v_\lambda\}_{\lambda\in\Omega}$ of L. Since f is injective, the $v'_\lambda := f(v_\lambda)$ are linearly independent. Extend this to a basis $\{v'_\mu\}_{\mu\in\Omega'}$ such that $\Omega \subset \Omega'$. Pick a total order on Ω'. This induces a total order on Ω. Then $M \in S\Omega$ implies $M \in S\Omega'$, and we have

$$U(f)(x_M) = x'_M$$

(Details?). Thus a basis gets mapped to a linearly independent set. □ ?

4.5 Induced representations

■ Let B be an algebra over k. Recall that "algebra" means "unital associative algebra". A *left module of B* is a k-vector space M together with a bilinear map $B \times M \to M$, $(b, m) \mapsto b.m$, such that for all $a, b \in B$, $m \in M$ we have

$$a.(b.m) = (ab).m \quad \text{and} \quad 1.m = m\ .$$

A *right module of B* is a k-vector space M together with a bilinear map $M \times B \to M$, $(m,b) \mapsto m.b$, such that for all $a,b \in B$, $m \in M$ we have

$$(m.a).b = m.(ab) \quad \text{and} \quad m.1 = m \ .$$

To indicate wether B acts from the left or from the right, we will sometimes write ${}_BM$ or M_B, respectively.

Definition 4.5.1. Let B be a k-algebra, M a right B-module and N a left B-module. The *relative tensor product* (or *tensor product over* B) of M and N is defined to be

$$M \otimes_B N \ := \ M \otimes N \,/\, \mathrm{span}_k\{m.b \otimes n - m \otimes b.n \,|\, b \in B, m \in M, n \in N\} \ ,$$

where $M \otimes N$ is the tensor product of the unterlying k-vector spaces.

■ Write $\pi : M \otimes N \to M \otimes_B N$ for the canonical projection and $m \otimes_B n := \pi(m \otimes n)$. Then by definition, for all $b \in B$, $m \in M$, $n \in N$,

$$(m.b) \otimes_B n = m \otimes_B (b.n) \ .$$

We call a bilinear map $\beta : M \times N \to V$ *B-balanced* if

$$\forall b \in B, m \in M, n \in N : \beta(m.b, n) = \beta(m, b.n) \ .$$

Thus by the above equality, the map $\otimes_B : M \times N \to M \otimes_B N$, $(m,n) \mapsto m \otimes_B n$ is B-balanced.

Lemma 4.5.2. Let B, M, N be as in the above definition. Then for each k-vector space V and each B-balanced bilinear map $\beta : M \times N \to V$, there exists a unique linear map $\widehat{\beta} : M \otimes_B N \to V$ such that $\beta = \widehat{\beta} \circ \otimes_B$:

$$\begin{tikzcd}
M \times N \arrow[r, "\otimes_B"] \arrow[d, "\forall\beta"'] & M \otimes_B N \arrow[dl, dashed, "\exists!\widehat{\beta}"] \\
A &
\end{tikzcd}$$

(The proof is an exercise.) E

■ Let B be a k-algebra, $A \subset B$ be a subalgebra, and M a left A-module. (Ex. 22)
The *induced representation of B on M* is

$$\mathrm{Ind}_A^B(M) := B \otimes_A M \ .$$

The B-action is given the next lemma:

Lemma 4.5.3. The map $c.(b \otimes_A m) := (cb) \otimes_A m$ $(b, c \in B,\ m \in M)$ is well-defined and turns $\mathrm{Ind}_A^B(M)$ into a B-module.

(The proof is an exercise.) E

(Ex. 23)

Definition 4.5.4. Let B be a k-algebra and M a right B-module. A family $\{m_\lambda\}_{\lambda\in\Lambda}$, $m_\lambda \in M$ is called a B-basis of M, if

1. for all $n \in M$ there are $b_\lambda \in B$ (only finitely many non-zero) such that $n = \sum_{\lambda\in\Lambda} m_\lambda.b_\lambda$, and
2. $\sum_{\lambda\in\Lambda} m_\lambda.b_\lambda = 0$ (finite sum) implies that all $b_\lambda = 0$.

If M has a B-basis, it is called *free*.

■ One can also formulate conditions 1 and 2 above as one condition by demanding that the b_λ which satisfy the equality in 1 are unique. For a left B-module, a B-basis and freeness are defined analogously.

Proposition 4.5.5. Let B be a k-algebra, M a right B-module and N a left B-module. Suppose M is free. Let $\{m_\alpha\}_{\alpha\in I}$ be a B-basis of M, and let $\{n_j\}_{j\in J}$ be a vector space basis of N. Then

$$\{m_\alpha \otimes_B n_j\}_{(\alpha,j)\in I\times J}$$

is a vector space basis of $M \otimes_B N$.

Proof. Abbreviate $S = \{m_\alpha \otimes_B n_j\}_{(\alpha,j)\in I\times J}$.

S spans $M \otimes_B N$: As $M \otimes N$ is spanned by $\{x \otimes y \,|\, x \in M, y \in N\}$ and $\pi : M \otimes N \to M \otimes_B N$ is surjective, $M \otimes N$ is spanned by $\{x \otimes_B y | x \in M, y \in N\}$. But every $x \otimes_B y$ already lies in the span of S:

$$x \otimes_B y \overset{(1)}{=} \sum_{\alpha\in I} m_\alpha.b_\alpha \otimes_B y \overset{(2)}{=} \sum_{\alpha\in I} m_\alpha \otimes_B b_\alpha.y \overset{(3)}{=} \sum_{\alpha\in I, j\in J} c_j\, m_\alpha \otimes_B n_j \ ,$$

where in step 1 we use that $\{m_\alpha\}$ is a B-basis, in step 2 that $\otimes_B$ is B-balanced, and in step 3 we expand $b_\alpha.y$ in the vector space basis of N as $b_\alpha.y = \sum_{j\in J} c_j n_j$. (24.11)

S is linearly independent in $M \otimes_B N$: Fix $\alpha \in I$ and define $\varphi_\alpha : M \to B$ via

$$\varphi_\alpha(m) = b_\alpha \qquad \text{for} \quad m = \sum_{\beta\in I} m_\beta.b_\beta \ .$$

This is well-defined by the uniqueness of the expansion in a B-basis. For the same reason, φ_α is an intertwiner of right B-modules: for all $b \in B$,

$$\varphi_\alpha(m.b) = \varphi_\alpha\Big(\sum_{\beta\in I} m_\beta.(b_\beta b)\Big) = b_\alpha b \ .$$

It follows that the map $f_\alpha : M \times N \to N$, $f(m,n) = \varphi_\alpha(m).n$ is bilinear and B-balanced (Details?), amd so by Lemma 4.5.2 there is a (unique) linear map $\widehat{f}_\alpha : M \otimes_B N \to N$ with the property ?

$$\widehat{f}_\alpha(m \otimes_B n) = \varphi_\alpha(m).n \ .$$

Let $\sum_{\beta\in I, j\in J} c_{\beta j} m_\beta \otimes_B n_j = 0$, $c_{\beta j} \in k$ be a decomposition of zero. Applying $\widehat{f}_\alpha$ results in

$$0 = \widehat{f}_\alpha\Big(\sum_{\beta\in I, j\in J} c_{\beta j}\, m_\beta \otimes_B n_j\Big) = \sum_{j\in J} c_{\alpha,j}\, n_j$$

(Why does the second equality hold?). But the n_j are linearly independent, and so $c_{\alpha,j} = 0$ for all j. ? $\square$

■ Let L be a Lie algebra over k. Given a vector space V, the L-representations $R : L \to gl(V)$ are in 1-1 correspondence to $U(L)$-representations via: (Exercise.) E

(Ex. 17)

$$\begin{tikzcd} L \arrow[r, "\iota_L"] \arrow[d, "\forall R"'] & U(L) \arrow[dl, dashed, "\exists!\widehat{R}"] \\ \mathrm{End}(V) & \end{tikzcd}$$

Let $H \subset L$ be a Lie subalgebra. Then also $U(H) \hookrightarrow U(L)$ is a subalgebra (Corollary 4.4.9). For an H-module M, we abbreviate

$$\mathrm{Ind}_H^L(M) := \mathrm{Ind}_{U(H)}^{U(L)}(M) = U(L) \otimes_{U(H)} M \ .$$

Proposition 4.5.6. Let $H \subset L$ be a Lie subalgebra. Then $U(L)$ is a free right $U(H)$-module.

Proof. Let $\{v_\lambda\}_{\lambda\in\Lambda}$ be a basis of H. Extend this basis to a basis $\{v_\mu\}_{\mu\in\Omega}$ of L, i.e. we have $\Lambda \subset \Omega$. Pick a total order on Ω with the property

$$\forall \mu \in \Omega \setminus \Lambda, \lambda \in \Lambda \ : \ \mu < \lambda \ . \qquad (*)$$

(Why is that always possible?) As before, we write $x_M = \iota_L(v_{\lambda_1}) \cdots \iota_L(v_{\lambda_m}) \in U(L)$ for $M = (\lambda_1, \dots, \lambda_m)$. The next claim shows the proposition. ?

Claim: $\{x_M\}_{M\in S(\Omega\setminus\Lambda)}$ is a $U(H)$-basis of $U(L)$.

⌈ Write $\Lambda^c := \Omega \setminus \Lambda$. Because of $(*)$ we have, for $M \in S\Lambda^c$ and $N \in S\Lambda$, that $x_M \cdot x_N = x_{MN}$, where "MN" stands for the sequence obtained by first taking the indices in M and then those in N.

In particular, this establishes property 1 in Definition 4.5.4: every $u \in U(L)$ can be written in the form $u = \sum_{M \in S\Lambda^c} x_M.h_M$ for some $h_M \in U(H)$, only finitely many nonzero (Why?). ?

To show property 2, suppose that $\sum_{M \in S\Lambda^c} x_M.h_M = 0$ for some choice of $h_M \in U(H)$. By the PBW-theorem, each h_M can be written as a linear combination

$$h_M = \sum_{N \in S\Lambda} c_M^N x_N \qquad \text{for some} \quad c_M^N \in k \ .$$

This gives the decomposition of zero

$$0 = \sum_{M \in S\Lambda^c} \sum_{N \in S\Lambda} c_M^N x_M x_N \ .$$

But the $x_M x_N = x_{MN}$ form a linearly independent set by the PBW-theorem, and hence all c_M^N must be zero. ⌋

□

Corollary 4.5.7. Let $H \subset L$ and consider bases $\{v_\lambda\}_{\lambda \in \Lambda}$ of H and $\{v_\mu\}_{\mu \in \Omega}$ as in the proof of Proposition 4.5.6. Let M be an H-module with vector space basis $\{v_i\}_{i \in I}$. Then a vector space basis of $\mathrm{Ind}_H^L(M)$ is given by

$$\left\{ x_M.v_i \,\middle|\, M \in S(\Omega \setminus \Lambda), i \in I \right\} \ .$$

Proof. The proof of Proposition 4.5.6 provides us with a $U(H)$-basis of $U(L)$. We can hence apply Proposition 4.5.5. □

Example 4.5.8 (Verma modules for $sl(2,k)$)**.** Consider the basis of $sl(2,k)$ introduced at the end of Section 2.1:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with ordering $f < h < e$. Let

$$b := \mathrm{span}_k\{h, e\} = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \,\middle|\, a, b \in k \right\} \subset sl(2,k) \ .$$

The b is a Lie subalgebra of $sl(2,k)$ which is solvable (Why?).

For $\lambda \in k$ let B_λ be the 1-dimensional representation of b given by $B_\lambda = k$ as a vector space, and with action on $1 \in k$ given by

$$e.1 = 0 \quad , \quad h.1 = \lambda\, 1 \ .$$

(Why is this a b-module? Why are these in fact all irreducible finite-dimensional b-modules, provided that k is algebraically closed and of characteristic zero?)

The *Verma module* V_λ of $sl(2,k)$ is defined to be

$$V_\lambda := \mathrm{Ind}_b^{sl(2,k)}(B_\lambda) \ .$$

By Corollary 4.5.7, V_λ has a vector space basis

$$\{u_m\}_{m \in \mathbb{Z}_{\geq 0}} \quad , \quad \text{where} \quad u_m = f^m \otimes_{U(b)} 1 \ ,$$

where $f^0 = 1$ and for $m > 0$, $f^m = f \cdots f$ is the m-fold product of f with itself in $U(sl(2,k))$ and $1 \in k = B_\lambda$ (Details?). Explicitly, the action on the first two basis elements is

$$\begin{aligned}
f.u_0 &= (f \cdot 1) \otimes_{U(b)} 1 = u_1 \ , \\
h.u_0 &= (h \cdot 1) \otimes_{U(b)} 1 = (1 \cdot h) \otimes_{U(b)} 1 = 1 \otimes_{U(b)} h.1 = \lambda\, u_0 \ , \\
e.u_0 &= e \otimes_{U(b)} 1 = 1 \otimes_{U(b)} e.1 = 0 \ , \\
f.u_1 &= ff \otimes_{U(b)} 1 = u_2 \ , \\
h.u_1 &= hf \otimes_{U(b)} 1 = [h,f] \otimes_{U(b)} 1 + fh \otimes_{U(b)} 1 \\
&= -2f \otimes_{U(b)} 1 + f \otimes_{U(b)} h.1 = (\lambda - 2) u_1 \ , \\
e.u_1 &= \lambda\, u_0 \ .
\end{aligned}$$

5 Semisimple Lie algebras

5.1 The Killing form and semisimplicity

Let k be a field.

Definition 5.1.1. Let U, V be k-vector spaces. A bilinear pairing b on $U \times V$ (i.e. a bilinear map $b : U \times V \to k$) is *non-degenerate* if

$$\big(\forall u \in U : b(u,v) = 0\big) \Rightarrow v = 0 \ \text{ and } \ \big(\forall v \in V : b(u,v) = 0\big) \Rightarrow u = 0 \ .$$

■ Let L be a finite-dimensional Lie algebra over k. Recall from Definition 3.3.3 and Remark 3.3.6 that the Killing form $\kappa = \kappa_L$ of L is given by $\kappa(x, y) = \mathrm{tr}_L(\mathrm{ad}(x)\,\mathrm{ad}(y))$ and satisfies

$$\kappa(x, y) = \kappa(y, x) \quad , \quad \kappa([x, y], z) = \kappa(x, [y, z]) \ .$$

For $sl(2, k)$ we saw in Example 3.3.4 (2) that

$$\kappa(h, h) = 8 \quad , \quad \kappa(e, f) = \kappa(f, e) = 4 \ ,$$

and all other pairs of basis vectors giving zero. Thus κ is non-degenerate iff $\mathrm{char}(k) \neq 2$.

Definition 5.1.2. Let L be a finite-dimensional Lie algebra over k, and let B be a non-degenerate symmetric invariant bilinear pairing on L. Let $\{x_1, \dots, x_n\}$ be a basis of L, and $\{y_1, \dots, y_n\}$ the dual basis with respect to B (i.e. $B(x_i, y_j) = \delta_{i,j}$). Then

$$C_B := \sum_{i=1}^{n} x_i y_i \ \in \ U(L)$$

is called the *Casimir element for B*.

(Check that C_B is independent of the choice of basis of L.) ?

Proposition 5.1.3. Let L and B be as in the previous definition. Then $C_B \in Z(U(L))$.

Proof. For all $a \in L$ we have

$$\sum_i B(a, x_i) y_i = a = \sum_i B(a, y_i) x_i \ .$$

(Why?) Using this, we compute for all $a \in L$?

$$\begin{aligned}[a, C_B] &= \sum_i [a, x_i y_i] \overset{(*)}{=} \sum_i [a, x_i] y_i + \sum_j x_j [a, y_j] \\ &= \sum_{i,j} B([a, x_i], y_j) x_j y_i + \sum_{i,j} x_j y_i B([a, y_j], x_i) \overset{(**)}{=} 0\end{aligned}$$

To see $(**)$ note that

$$B([a, y_j], x_i) \overset{\text{sym.}}{=} B(x_i, [a, y_j]) \overset{\text{inv.}}{=} B([x_i, a], y_j) = -B([a, x_i], y_j) \ .$$

(Why does $(*)$ hold?) □ ?

■ If the Killing form κ of L is non-degenerate, then $C = C_\kappa$ is called *the Casimir element of L.*

(28.11)

■ Recall from Remark 3.3.6 the notation

$$V^\perp = \{a \in L \,|\, \kappa_L(a,v) = 0 \text{ for all } v \in V\}$$

for a subset $V \subset L$. For example κ_L is non-degenerate iff $L^\perp = \{0\}$.

Lemma 5.1.4. Let L be a finite-dimensional Lie algebra over k, and let $I \subset L$ be an ideal. Then:

1. $I^\perp$ is an ideal of L.
2. If I is in addition abelian, then $I \subset L^\perp$.
3. If κ_L is non-degenerate, then $L = I \oplus I^\perp$ as Lie algebras.

Proof. Part 1: Follows from symmetry of κ_L. (Details?) ?

Part 2: We may assume that $I \neq \{0\}$. Let $\{x_1, \dots, x_{\dim I}\}$ be a basis of I. Extend this to a basis $\{x_1, \dots, x_{\dim L}\}$ of L. Write x_j^* for the dual basis of L^*. For all $y \in L$ and $i \in I$ we have:

$$\kappa_L(y,i) = \sum_{a=1}^{\dim I} x_a^*\big([y, \underbrace{[i,x_a]}_{=0 \text{ as } I \text{ abelian}}]\big)] + \sum_{a=\dim I+1}^{\dim L} x_a^*\big(\underbrace{[y,[i,x_a]]}_{\in I \text{ as } I \text{ ideal}}\big) = 0 \ .$$

Part 3: We will show $I \cap I^\perp = \{0\}$ and $\dim I + \dim I^\perp = \dim L$. This implies part 3.

■ $I \cap I^\perp = \{0\}$: Let $A := I \cap I^\perp$.

Claim: A is an abelian ideal of L.

> $\lceil$ Clearly A is an ideal. To see that it is abelian, let $x \in L$ and $a, b \in A$. Then
>
> $$\kappa_L(x,[a,b]) = \kappa_L([x,a],b) = 0 \ ,$$
>
> since $[x,a] \in A \subset I$ and $b \in A \subset I^\perp$. As κ_L is non-degenerate, it follows that $[a,b] = 0$. $\rfloor$

By part 2, $A \subset L^\perp$. But $L^\perp = \{0\}$ as κ_L is non-degenerate.

■ $\dim I + \dim I^\perp = \dim L$: Consider the composition of linear maps

$$f : L \longrightarrow L^* \longrightarrow I^* \quad , \quad x \mapsto \kappa_L(x,-) \mapsto \kappa_L(x,-)|_I \ .$$

The second map is restricting the arguments from L to I, and is surjective (Why?). By definition, the kernel of f is $I^\perp$. Thus we have ?

$$\dim L = \dim(\mathrm{im}(f)) + \dim(\ker(f)) = \dim(I^\perp) + \dim(I^*) \ .$$

But $\dim(I^*) = \dim(I)$. □

Lemma 5.1.5. Let L be a Lie algebra over k. Suppose $L = \bigoplus_{\alpha\in A} L_\alpha$ as Lie algebras, where all L_α are simple Lie algebras. Let $I \subset L$ be an ideal. Then there is a subset $B \subset A$ such that $I = \bigoplus_{\beta\in B} L_\beta$.

■ The lemma also implies that the direct summands in the decompostion $L = \bigoplus_{\alpha\in A} L_\alpha$ are unique up to permutation. (Why?) ?

Proof. Let $i \in I$ be arbitrary. We can write (uniquely) $i = \sum_{\alpha\in A} x_\alpha$ with $x_\alpha \in L_\alpha$. We will show

$$x_\alpha \neq 0 \ \Rightarrow \ L_\alpha \subset I \ .$$

This implies the claim. (Why? What is B?) ?

Suppose thus that $x_\alpha \neq 0$. Define

$$M_\alpha := [L_\alpha, i] + [L_\alpha, [L_\alpha, i]] + [L_\alpha, [L_\alpha, [L_\alpha, i]]] + \cdots \ .$$

Since $[L_\alpha, i] = [L_\alpha, x_\alpha] \subset L_\alpha$, etc., all of the above summands are contained in L_α. Since $i \in I$ and I is an ideal, all summands are also contained in I. By construction M_α satisfies $[L_\alpha, M_\alpha] \subset M_\alpha$ (Why?), i.e. it is an ideal of L_α. ?

We have $[L_\alpha, x_\alpha] \neq \{0\}$, for else the 1-dimensional span of x_α would be an ideal in L_α, but L_α is simple, and so we would have $L_\alpha = kx_\alpha$ (Why would that be a problem?). Hence $M_\alpha \neq \{0\}$. Since M_α is a non-zero ideal in L_α, we must have $M_\alpha = L_\alpha$. Since $M_\alpha \subset I$, it follows that $L_\alpha \subset I$. □ ?

■ Let L be a finite-dimensional Lie algebra over k. Recall from Definition 3.2.6 that $\mathrm{rad}(L)$ is the maximal solvable ideal of L, and that L is called semisimple if $\mathrm{rad}(L) = \{0\}$.

Theorem 5.1.6. Let L be a finite-dimensional Lie algebra over an algebraically closed field k of characteristic zero. The following are equivalent:

1. L is semisimple.
2. κ_L is non-degenerate.
3. $L = \bigoplus_{\alpha\in A} L_\alpha$ with L_α simple.

■ We will show 2 ⇒ 3 ⇒ 1 ⇒ 2. Of these, 2 ⇒ 3 ⇒ 1 work over a general field, and only 1 ⇒ 2 uses the assumptions $\mathrm{char}(k) = 0$ and $\overline{k} = k$. We will make use of this in the next section when we prove the above theorem also over $\mathbb{R}$ (cf. Theorem 5.2.8 below).
For info: In fact the above theorem holds for fields of characteristic zero, algebraic closedness is not required, see the comment after Theorem 5.2.8 below.

Proof. For $L = \{0\}$ the statement is trivial. Assume that $L \neq \{0\}$.

2 ⇒ 3 (this implication holds for a general field k): Let $\{0\} \subsetneq I \subsetneq L$ be an ideal. If such an ideal does not exist, then L is already simple (How can the case that L is abelian be excluded?). By Lemma 5.1.4 (3), $L = I \oplus I^\perp$ as Lie algebras. ?

Since $\kappa_L(i, j) = 0$ for all $i \in I$ and $j \in I^\perp$, we have that κ_I and $\kappa_{I^\perp}$ are both non-degenerate. (Why? *Hint:* Use that κ_L is non-degenerate, together with Remark 3.3.6 (2).) We can thus iterate the above process for I and $I^\perp$, decomposing into further direct summands until every summand simple. This process terminates as L is finite-dimensional. ?

3 ⇒ 1 (this implication holds for a general field k): $\mathrm{rad}(L) = \{0\}$ if and only if L has no non-zero abelian ideas (Exercise). Suppose thus that $I \subset L$ is an abelian ideal. By Lemma 5.1.5, $I = \bigoplus_{\beta \in B} L_\beta$ for some $B \subset A$. But none of the L_α is abelian, so we must have $B = \emptyset$. E (Ex. 24)

1 ⇒ 2: Let $I = L^\perp$. We need to show that $I = \{0\}$. Note that by definition $\kappa_L(i, x) = 0$ for all $i \in I$, $x \in L$ and so in particular

$$\kappa_L(i, j) = 0 \quad \text{for all} \quad i \in I, j \in [I, I] \ .$$

By Remark 3.3.6 (2), $\kappa_L = \kappa_I$, and we can apply Cartan's criterion (in the form of Corollary 3.3.5) to I. This shows that I is solvable, and hence $I \subset \mathrm{rad}(L)$. But $\mathrm{rad}(L) = \{0\}$ by assumption. □

Remark 5.1.7.
1. Recall from Example 2.2.3 (2) that $\mathrm{ad} : L \to \mathrm{Der}(L) \subset gl(L)$ is a Lie algebra homomorphism. Derivations in the image of ad are called *inner derivations*.
2. For a semisimple Lie algebra L we have $Z(L) \subset \mathrm{Rad}(L) = \{0\}$, and so $\mathrm{ad} : L \to \mathrm{Der}(L)$ is injective.

Proposition 5.1.8. For a finite-dimensional semisimple Lie algebra over an algebraically closed field, every derivation is inner.

Proof. Define a bilinear form b on $\mathrm{Der}(L)$ by setting $b(D, D') := \mathrm{tr}_L(D \circ D')$. Note that for $x, y \in L$ we have $\kappa(x, y) = b(\mathrm{ad}(x), \mathrm{ad}(y))$, and in this sense b extends the Killing form to include derivations. As for κ one checks that b is invariant:

$$b([D, D'], D'') = b(D, [D', D'']) \quad \text{for all } D, D', D'' \in \mathrm{Der}(L) .$$

Define

$$I = \mathrm{ad}(L)^{\perp} = \{D \in \mathrm{Der}(L) \,|\, b(D, \mathrm{ad}(x)) = 0 \text{ for all } x \in L\} .$$

We have $I + \mathrm{ad}(L) = \mathrm{Der}(L)$ (Why? *Hint:* Pick a dual basis pair of L with respect to κ, i.e. $\kappa(x_i, y_j) = \delta_{i,j}$. For $D \in \mathrm{Der}(L)$ consider $D' = D - \sum_i b(D, \mathrm{ad}(x_i))\,\mathrm{ad}(y_i)$ and show that $D' \in I$.) Since b is invariant, I is an ideal in the Lie algebra $\mathrm{Der}(L)$. Since b is non-degenerate on $\mathrm{ad}(L)$ (by non-degeneracy of the Killing form, see Theorem 5.1.6), $I \cap \mathrm{ad}(L) = \{0\}$. ?

So far we have shown $\mathrm{Der}(L) = I \oplus \mathrm{ad}(L)$. To complete the proof we need to show $I = \{0\}$. Let thus $D \in I$. We compute, for any $x \in L$,

$$\mathrm{ad}(D(x)) \overset{(1)}{=} [D, \mathrm{ad}(x)] \overset{(2)}{=} 0 ,$$

Step 1 follows as D is a derivation (Details?). In step 2 we use that I is an ideal, so that $[D, \mathrm{ad}(x)] \in I$, while the initial expression is clearly in $\mathrm{ad}(L)$. But $I \cap \mathrm{ad}(L) = \{0\}$. ?

Finally, by Remark 5.1.7 (2), ad is injective, and hence $D(x) = 0$ for all $x \in L$. □

Remark 5.1.9.

1. The proof of the above proposition relied on non-degeneracy of κ_L. This also holds over $\mathbb{R}$ (cf. Theorem 5.2.8 below), and so also for a finite-dimensional semisimple Lie algebra over $\mathbb{R}$, all derivations are inner.

2. For info: Let G be a Lie group, and let $\phi \in \mathrm{Aut}(G)$ a Lie group automorphism of G. Since $\phi(e) = e$, the differential $d\phi$ is a linear isomorphism $T_eG \to T_eG$. Recall that we can define the Lie algebra of G to be $g = T_eG$. One finds that $d\phi$ is a Lie algebra automorphism of g. We obtain a group homomorphism

$$D : \mathrm{Aut}(G) \to \mathrm{Aut}(g) .$$

Let $\phi_t \in \mathrm{Aut}(G)$, $t \in (-\varepsilon, \varepsilon) \subset \mathbb{R}$ be a family of automorphisms with $\phi_0 = \mathrm{id}$, such that $t \mapsto D \circ \phi_t$ is differentiable. Then $\frac{d}{dt} D \circ \phi_t|_{t=0}$ defines a derivation on g.

An automorphism $\phi \in \mathrm{Aut}(G)$ is called *inner* of $\phi(x) = axa^{-1}$ for some $a \in G$. If a family ϕ_t as above consists of inner automorphisms, the corresponding derivation is also inner.

Consider the homomorphism of Lie groups

$$\mathrm{Ad} : G \to \mathrm{Aut}(g) \quad , \quad a \mapsto D(a(-)a^{-1}) \ .$$

If the Lie algebra g of G is semisimple, then all derivations are inner, and one can show that $\mathrm{Aut}(g)/\mathrm{Ad}(G)$ is discrete. See e.g. Knapp, Lie groups beyond an introduction, Section I.14, and Kirillov, Proposition 6.8 for more.

5.2 Complexification of real Lie algebras

From here on we will only work with Lie algebras over $\mathbb{R}$ or over $\mathbb{C}$.

Definition 5.2.1. Let g be a Lie algebra over $\mathbb{R}$. The *complexification $g_{\mathbb{C}}$ of g* is $g_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} g$.

Remark 5.2.2. The following two descriptions of the Lie bracket on $g_{\mathbb{C}}$ are equivalent (Why?): ?

1. We can write $g_{\mathbb{C}} = g \oplus ig$, where we identify a in the first copy of g with $1 \otimes_{\mathbb{R}} a \in g_{\mathbb{C}}$, and b in the second copy with $i(1 \otimes_{\mathbb{R}} b) = i \otimes_{\mathbb{R}} b \in g_{\mathbb{C}}$. Thus each $x \in g_{\mathbb{C}}$ can be uniquely written as $x = a + ib$. Let further $y = c + id$. Then

$$[x, y]_{g_{\mathbb{C}}} := [a, c]_g - [b, d]_g + i\big([a, d]_g + [b, c]_g\big) \ .$$

2. For $\lambda, \mu \in \mathbb{C}$ and $a, b \in g$ we define

$$[\lambda \otimes_{\mathbb{R}} a, \mu \otimes_{\mathbb{R}} b]_{g_{\mathbb{C}}} := (\lambda\mu) \otimes_{\mathbb{R}} [a, b]_g \ .$$

Definition 5.2.3. A Lie algebra g over $\mathbb{R}$ is called a *real form* of a Lie algebra L over $\mathbb{C}$ if $L \cong g_{\mathbb{C}}$.

Remark 5.2.4. 1. Let $\{x_\alpha\}_{\alpha \in A}$ be a basis of a Lie algebra L over a field k. We can expand the Lie bracket of two basis vectors as

$$[x_\alpha, x_\beta] = \sum_{\gamma \in A} f_{\alpha\beta}{}^{\gamma}\, x_\gamma \quad \text{for some} \quad f_{\alpha\beta}{}^{\gamma} \in k \ .$$

The numbers $f_{\alpha\beta}{}^{\gamma}$ are called *structure constants* of L. They depend on L and on a choice of basis for L, and they determine the Lie bracket uniquely. (Since the structure constants determine the Lie bracket, one can express the defining conditions of a Lie algebra in terms of structure constants. Can you write out these conditions?) ?

2. Let $\{x_\alpha\}_{\alpha \in A}$ be a basis of g over $\mathbb{R}$. By Proposition 4.5.5 (for $B = \mathbb{R}$, $M = \mathbb{C}$, which is free over $\mathbb{R}$, and $N = g$), the set $\{1 \otimes_{\mathbb{R}} x_\alpha, i \otimes_{\mathbb{R}} x_\alpha\}_{\alpha \in A}$ is an $\mathbb{R}$-basis of $g_{\mathbb{C}}$, and so just $\{1 \otimes_{\mathbb{R}} x_\alpha\}_{\alpha \in A}$ is a $\mathbb{C}$-basis of $g_{\mathbb{C}}$ (Why?). ?

Claim: Let g be a real Lie algebra and L a complex Lie algebra. The following are equivalent:

(a) $L \cong g_{\mathbb{C}}$ as Lie algebras over $\mathbb{C}$.

(b) There exists a basis $\{x_\alpha\}_{\alpha \in A}$ of g over $\mathbb{R}$, and a basis $\{\tilde{x}_\alpha\}_{\alpha \in A}$ of L over $\mathbb{C}$ (with the same indexing set A), such that

$$[x_\alpha, x_\beta] = \sum_{\gamma \in A} f_{\alpha\beta}{}^{\gamma}\, x_\gamma \quad \text{and} \quad [\tilde{x}_\alpha, \tilde{x}_\beta] = \sum_{\gamma \in A} f_{\alpha\beta}{}^{\gamma}\, \tilde{x}_\gamma \ ,$$

with the same set of structure constants for both g and L.

⌈ For $(a) \Rightarrow (b)$, let $\varphi : g_{\mathbb{C}} \to L$ be the $\mathbb{C}$-linear Lie algebra homomorphism. Pick an $\mathbb{R}$-basis $\{x_\alpha\}_{\alpha \in A}$ and set $\tilde{x}_\alpha := \varphi(1 \otimes_{\mathbb{R}} x_\alpha)$. One checks that the structure constants agree (Details?). ?

For $(b) \Rightarrow (a)$, define the $\mathbb{C}$-linear isomorphism $\varphi : g_{\mathbb{C}} \to L$ on a $\mathbb{C}$-basis via $\varphi(1 \otimes_{\mathbb{R}} x_\alpha) = \tilde{x}_\alpha$. One checks that this is a Lie algebra homomorphism (Details?). ⌋ ?

(1.12.)

Example 5.2.5. 1. $su(2)_{\mathbb{C}} \cong sl(2, \mathbb{C})$: Recall:

$$su(2) = \{M \in \mathrm{Mat}(2, \mathbb{C}) \mid \mathrm{tr}(M) = 0, M^\dagger + M = 0\} \quad \text{(Lie algebra over } \mathbb{R})$$
$$sl(2, \mathbb{C}) = \{M \in \mathrm{Mat}(2, \mathbb{C}) \mid \mathrm{tr}(M) = 0\} \quad \text{(Lie algebra over } \mathbb{C})$$

We will show in two ways that $su(2)$ is a real form of $sl(2, \mathbb{C})$, i.e. that $su(2)_{\mathbb{C}} \cong sl(2, \mathbb{C})$ as complex Lie algebras:

- *Via a basis:* We check the condition in Remark 5.2.4 (2 b). Recall the $\mathbb{R}$-basis of $su(2)$ in terms of Pauli matrices at the end of Section 1.6,

$$x_1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \ , \quad x_2 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \quad x_3 = i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \ .$$

Note that $\widetilde{x}_1 := x_1$, $\widetilde{x}_2 := x_2$, $\widetilde{x}_3 := x_1$ is a $\mathbb{C}$-basis of $sl(2, \mathbb{C})$. As the matrices are the same, so are the structure constants.

- *Without basis:* By definition $su(2) \subset sl(2,\mathbb{C})$. The map

$$\varphi : su(2)_\mathbb{C} \to sl(2,\mathbb{C}) \quad , \quad 1 \otimes_\mathbb{R} A + i \otimes_\mathbb{R} B \mapsto A + iB \ ,$$

where $A, B \in su(2)$, is $\mathbb{C}$-linear and a Lie algebra isomorphism. (Why? *Hint:* If $A + iB = 0$, then also $(A+iB) \pm (A+iB)^\dagger = 0$, and if $\operatorname{tr} M = 0$, then $M - M^\dagger$ and $i(M + M^\dagger)$ are in $su(2)$.) ?

2. $sl(2,\mathbb{R})_\mathbb{C} \cong sl(2,\mathbb{C})$: The same expression for φ as above gives an isomorphism $sl(2,\mathbb{R})_\mathbb{C} \to sl(2,\mathbb{C})$ of complex Lie algebras. (Details?) ?

3. Parts 1 and 2 show that $sl(2,\mathbb{R})_\mathbb{C} \cong su(2)_\mathbb{C}$ as complex Lie algebras. But $sl(2,\mathbb{R})$ and $su(2)$ are *not* isomorphic as real Lie algebras (Exercise). E (Ex. 26)

4. $sl(2,\mathbb{C})_\mathbb{C} \cong sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})$: Consider the map

$$\varphi : sl(2,\mathbb{C})_\mathbb{C} \to sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C}) \ , \ 1 \otimes_\mathbb{R} A + i \otimes_\mathbb{R} B \mapsto \big(A + iB, \ \overline{A} + i\overline{B}\big) \ ,$$

where $A, B \in sl(2,\mathbb{C})$ and $\overline{A}$ stands for the matrix obtained by complex conjugating all entries in A, i.e. $\overline{A} = (A^t)^\dagger$. The map φ is $\mathbb{C}$-linear, bijective, and a Lie algebra homomorphism. (Why? What would go wrong if we would take $\overline{A} - i\overline{B}$ in the second summand in the definition of φ?) ?

Note the notational pitfall that e.g.

$$i \otimes_\mathbb{R} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq 1 \otimes_\mathbb{R} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{in} \quad sl(2,\mathbb{C})_\mathbb{C} \ ,$$

and so one should not abbreviate $i \otimes_\mathbb{R} M$ as iM.

5. Further isomorphisms of complex Lie algebras are (Exercise): E (Ex. 28, 32)

$$su(n)_\mathbb{C} \cong sl(n,\mathbb{C}) \quad , \quad u(n)_\mathbb{C} \cong gl(n,\mathbb{C}) \quad , \quad so(p,q)_\mathbb{C} \cong so(p+q,\mathbb{C}) \quad ,$$

For info: The Lie algebra of the 4-dimensional Lorentz group (recall Section 1.2 and Example 1.5.7) satisfies $so(1,3) \cong sl(2,\mathbb{C})$ as *real* Lie algebras, and thus $so(1,3)_\mathbb{C} \cong sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})$ as complex Lie algebras. In 4d QFT one can label particles by pairs of $sl(2,\mathbb{C})$ representations, see e.g. Weinberg, Quantum theory of fields vol. 1, Section 5.6.

Lemma 5.2.6. Let g be a Lie algebra over $\mathbb{R}$. Then g is nilpotent (resp. solvable) iff $g_\mathbb{C}$ is nilpotent (resp. solvable).

Proof. *Nilpotent:* In terms of elements, the statement that a Lie algebra L over a field k is nilpotent reads (cf. Definition 3.1.1): There exists an N such that for all $x_1, \dots, x_N \in L$ we have

$$[x_1, [x_2, \cdots [x_{N-2}, [x_{N-1}, x_N]] \cdots]] = 0 \ .$$

Writing an element $z_j \in g_{\mathbb{C}}$ as $z_j = 1 \otimes_{\mathbb{R}} a_j + i \cdot 1 \otimes_{\mathbb{R}} b_j$ for $a_j, b_j \in g$ shows that the above statement specialised to g is equivalent to that for $g_{\mathbb{C}}$.

Solvable: Analogous. (Details?) □ ?

Remark 5.2.7. Let g be a finite-dimensinoal Lie algebra over $\mathbb{R}$. The Killing forms of g and $g_{\mathbb{C}}$ are related as follows: For all $x, y \in g$, $\lambda, \mu \in \mathbb{C}$,

$$\kappa_{g_{\mathbb{C}}}(\lambda \otimes_{\mathbb{R}} x, \mu \otimes_{\mathbb{R}} y) \ = \ \lambda\mu \cdot \kappa_g(x, y) \ .$$

In particular $\kappa_{g_{\mathbb{C}}}$ is non-degenerate iff κ_g is non-degenerate. (Why does the formular for $\kappa_{g_{\mathbb{C}}}$ hold and why does this give the claimed equivalence?) ?

We get the following variant of Theorem 5.1.6 for $\mathbb{R}$, which holds even though $\mathbb{R}$ is not algebraically closed:

Theorem 5.2.8. Let L be a finite-dimensional Lie algebra over $\mathbb{R}$. The following are equivalent:

1. L is semisimple.

2. κ_L is non-degenerate.

3. $L = \bigoplus_{\alpha \in A} L_\alpha$ with L_α simple.

For info: More generally, the above results holds for L a finite-dimensional Lie algebra over a field k of characteristic zero. In the proof one needs to substitute the algebraic closure $\overline{k}$ of k whenever $\mathbb{C}$ is used in place of $\mathbb{R}$.

Proof. As noted in the proof of Theorem 5.1.6, it only remains to show $1 \Rightarrow 2$. And as we did there, we set $I = L^\perp$ and observe that

$$\kappa_I(i, j) = 0 \quad \text{for all} \quad i \in I, j \in [I, I] \ .$$

Consider now the complexification $I_{\mathbb{C}}$ of the ideal I, seen as a Lie algebra of its own. Together with Remark 5.2.7, the above identity implies that also (Why?) ?

$$\kappa_{I_{\mathbb{C}}}(a, b) = 0 \quad \text{for all} \quad a \in I_{\mathbb{C}}, b \in [I_{\mathbb{C}}, I_{\mathbb{C}}] \ .$$

Cartan's criterion (in the form of Corollary 3.3.5) applied to $I_{\mathbb{C}}$ shows that $I_{\mathbb{C}}$ is solvable. By Lemma 5.2.6, also I is solvable. Hence $I \subset \mathrm{rad}(L) = \{0\}$. □

Combining Remark 5.2.7 and condition 2 of Theorems 5.1.6 and 5.2.8, we obtain:

Corollary 5.2.9. Let L be a finite-dimensional Lie algebra over $\mathbb{R}$. Then L is semisimple iff $L_{\mathbb{C}}$ is semisimple.

Remark 5.2.10. Note that L simple does not imply $L_{\mathbb{C}}$ simple, see Example 5.2.5 (4). However, if $L_{\mathbb{C}}$ is simple, then so is L. (Why? *Hint:* Suppose $I \subset L$ is a non-trivial ideal ...) ?

Example 5.2.11.

1. $su(n)$ *is semisimple:* By Example 5.2.5 (5) we have $su(n)_{\mathbb{C}} \cong sl(n, \mathbb{C})$, and the latter is simple (Exercise) and in particular semisimple (by condition 3 in Theorem 5.1.6), and so by Corollary 5.2.9, $su(n)$ is semisimple. In fact, by Remark 5.2.10, $su(n)$ is even simple. E (Ex. 8)

2. $u(n)$ *is not semisimple:* By Example 2.3.7, $u(n) \cong u(1) \oplus su(n)$, but $u(1)$ is an abelian ideal, and so $\mathrm{rad}(u(n)) \neq \{0\}$ (Exercise). E (Ex. 24)

■ For info: Let g be the Lie algebra (over $\mathbb{R}$) of a compact Lie group G (i.e. the manifold G is compact as a topological space). Then $g = g_{ss} \oplus g_{ab}$, where g_{ss} is semisimple and g_{ab} is abelian. See Knapp, Lie groups beyond an introduction (Birkhäuser 2002), Corollary 4.25. In the above example, both $SU(n)$ and $U(n)$ are compact Lie groups.

5.3 Complete reducibility

Definition 5.3.1. A representation V of a Lie algebra L over a field k is *completely reducible* (or *semisimple*) if

$$V = \bigoplus_{\alpha \in A} V_\alpha ,$$

where each $V_\alpha \subset V$ is an irreducible subrepresentation of V.

Theorem 5.3.2 (Weyl's Theorem)**.** A finite-dimensional representation of a finite-dimensional semisimple complex Lie algebra is completely reducible.

Definition 5.3.3. Let L be a Lie algebra over a field k. A short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

of L-modules and L-module intertwiners *splits* if there is an L-module isomorphism $\phi : B \to A \oplus C$ such that

$$\begin{tikzcd}
& B \arrow[dr, "g"] \arrow[dd, "\phi"] & \\
A \arrow[ur, "f"] \arrow[dr, "\iota_1"'] & & C \\
& A \oplus C \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

commutes. Here, $\iota_1(a) = a \oplus 0$ and $\pi_2(a \oplus c) = c$.

■ The sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of L-modules is called *exact* if f, g are L-module intertwiners such that f is injective, g is surjective, and $\mathrm{im}(f) = \ker(g)$ holds in B. One can have exact seqences of arbitrary length. The ones we will use have three terms and are also called *short exact sequences*, often abbreivated as $A \xrightarrow{f} B \xrightarrow{g} C$.

Lemma 5.3.4. With the notation of Definition 5.3.3: The following are equivalent:

1. The exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits.

2. There is an L-module intertwiner $A \xleftarrow{r} B$ such that $[A \xleftarrow{r} B \xleftarrow{f} A] = \mathrm{id}_A$ (i.e. $r \circ f = \mathrm{id}_A$).

3. There is an L-module intertwiner $B \xleftarrow{s} C$ such that $[C \xleftarrow{g} B \xleftarrow{s} C] = \mathrm{id}_C$ (i.e. $g \circ s = \mathrm{id}_C$).

Proof. First note that $1 \Rightarrow 2$ and $1 \Rightarrow 3$ are immediate. For example, with $\pi_1 : A \oplus C \to A$ the projection onto the first component, we can set $r = \pi_1 \circ \phi : B \to A$. (Why does that satisfy $r \circ f = \mathrm{id}_A$? What does one ?
choose for s?)

For $2 \Rightarrow 1$, we make the ansatz

$$\phi : B \to A \oplus C \quad , \quad b \mapsto (r(b), g(b))$$

and check its properties:

- *ϕ makes the diagram commute:* By definition, $\pi_2(\phi(b)) = g(b)$. Furthermore, $\phi(f(a)) = (r(f(a)), g(f(a)) = (a, 0) = \iota_1(a)$.

- *ϕ is surjective:* Let $(a,c) \in A \oplus C$. Since g is surjective, there is $\tilde{b}$ such that $g(\tilde{b}) = c$. Define

$$b := \tilde{b} - f(r(\tilde{b})) + f(a) \ .$$

Then $r(b) = r(\tilde{b}) - r(f(r(\tilde{b}))) + r(f(a)) = r(\tilde{b}) - r(\tilde{b})) + a = a$ and $g(b) = g(\tilde{b}) - g(f(r(\tilde{b}))) + g(f(a)) = c + 0 + 0$, so that altogether $\phi(b) = (a,c)$.

- *ϕ is injective:* Suppose $\phi(b) = 0$ for some $b \in B$. Then $g(b) = 0$ and by exactness, there is an $a \in A$ such that $f(a) = b$. But then $0 = \phi(b) = \phi(f(a)) = (a, 0)$ and so $a = 0$ and hence $b = 0$.

The arguments for $3 \Rightarrow 1$ are analogous. (Details?) □ ?

■ Below we often use the abbreviation "fssc Lie algebra" for finite dimensional semisimple complex Lie algebra. (5.12.)

Lemma 5.3.5. Let L be a fssc Lie algebra and (V,R) a finite dimensional faithful (i.e. $R : L \to gl(V)$ is injective) representation of L. For $x, y \in L$ set $B(x,y) = \mathrm{tr}_V(R(x)R(y))$. Then

1. B is non-degenerate, invariant, symmetric.

2. $\mathrm{tr}_V\, R(C_B) = \dim(L)$, with C_B as in Definition 5.1.2.

Proof. Part 1: Invariance and symmetry are check in the same way as for κ_L in Remark 3.3.6 (1) (Details?). For non-degeneracy we will use Cartan's criterion. Let $K = \{z \in L \,|\, B(z,x) = 0 \text{ for all } x \in L\}$. Then K is an ideal in L (Why?). Set $\widetilde{K} = R(K) \subset gl(V)$. Then ? ?

$$\mathrm{tr}_V(zy) = 0 \quad \text{for all} \quad z \in \widetilde{K} \ , \quad y \in [\widetilde{K}, \widetilde{K}]$$

(this would be true by definition even for $y \in R(L)$). By Cartan's criterion (Theorem 3.3.1), $\widetilde{K}$ is solvable. As R is injective, and hence $R|_K \to \widetilde{K}$ is a Lie algebra isomorphism, also K is solvable. Altogether, K is a solvable ideal in L, and hence $K \subset \mathrm{rad}(L) = \{0\}$ (as L is semisimple).

Part 2: We use bases $\{x_i\}$ and $\{y_i\}$ of L such that $B(x_i, y_j) = \delta_{i,j}$ as in Definition 5.1.2. Then

$$\mathrm{tr}_V\, R(C_B) = \sum_{i=1}^{\dim L} \mathrm{tr}_V \underbrace{R(x_i y_i)}_{=R(x_i)R(y_i)} = \sum_{i=1}^{\dim L} \underbrace{B(x_i, y_i)}_{=1} = \dim(L) \ ,$$

where by abuse of notation we also wrote $R : U(L) \to \mathrm{End}(V)$ for the representation of the algebra $U(L)$ corresponding to the representation $R : L \to gl(V)$ of the Lie algebra L. □

Lemma 5.3.6. Let L be a fssc Lie algebra and (V,R) a finite dimensional representation of L. Then $\mathrm{im}(R) \subset sl(V)$, i.e. $\mathrm{tr}_V(R(x)) = 0$ for all $x \in L$.

Proof. Since L is semisimple, we have $[L,L] = L$ (Why? *Hint:* For simple Lie algebras you can look at the hint below Definition 3.2.6). In other words, L is spanned by elements of the form $[x,y]$ with $x,y \in L$. But ?

$$\mathrm{tr}_V(R([x,y])) = \mathrm{tr}_V([R(x),R(y)]) = 0 \ .$$

□

Corollary 5.3.7. A fssc Lie algebra acts as zero on a 1-dimensional representation.

(Why?) ?

Proof of Weyl's Theorem (Theorem 5.3.2). We will show that every short exact sequence

$$0 \to W \xrightarrow{\iota} V \xrightarrow{\pi} Q \to 0$$

of L-modules splits (Why does that prove the theorem?). We will identify W with its image $\iota(W)$ in V. ?

Step 1. Assume $\dim(Q) = 1$ *and* W *irreducible:*

Claim: For all $x \in L$, $v \in V$, we have $x.v \in W$.

⌈ Since $\pi : V \to Q$ is an L-module intertwiner, we have $\pi(x.v) = x.\pi(v)$. As Q is 1-dimensional, Corollary 5.3.7 gives $x.\pi(v) = 0$. Thus $x.v \in \ker(\pi) = \mathrm{im}(\iota) = W$. ⌋

Let $R : L \to gl(V)$ denote the action of L on V. If $R = 0$, there is nothing to do: any linear map $r : V \to W$ such that $r|_W = \mathrm{id}_W$ gives a splitting as the condition of being an L-module intertwiner is empty in this case.

Suppose therefore $R \neq 0$: Define $\widetilde{L} = R(L) \subset gl(V)$, and denote the embedding $\widetilde{L} \hookrightarrow gl(V)$ by $\widetilde{R}$. For $x,y \in \widetilde{L}$ set $B(x,y) = \mathrm{tr}_V(\widetilde{R}(x)\widetilde{R}(y))$ and define $C_B \in Z(U(\widetilde{L}))$ as in Lemma 5.3.5, applied to $\widetilde{L}$ and $(V,\widetilde{R})$. The above claim implies (How?) ?

$$\widetilde{R}(C_B)v \in W \quad \text{for all} \quad v \in V \ . \qquad (*)$$

Since $W \subset V$ is irreducible as an L-module, it is also irreducible as an $\widetilde{L}$-module, and so by Schur's Lemma (Theorem 2.2.8),

$$\widetilde{R}(C_B)|_W = \lambda\, \mathrm{id}_W \qquad \text{for some} \quad \lambda \in \mathbb{C} \ . \qquad (**)$$

Claim: $\lambda \neq 0$.

⌈ Pick a basis of W and extend to a basis of V. By $(*)$, the matrix representing $\widetilde{R}(C_B)$ is of the form

$$\left(\begin{array}{ccc|c} \lambda & & 0 & \\ & \ddots & & * \\ 0 & & \lambda & \\ \hline & 0 & & 0 \end{array}\right)$$

and so the trace over V is given by $\lambda \dim(W)$. By Lemma 5.3.5, $\mathrm{tr}_V \widetilde{R}(C_B) = \dim \widetilde{L} > 0$, and so we must have $\lambda \neq 0$. ⌋

Because of $(*)$ and the above claim, we can define $r : V \to W$, $r(v) = \frac{1}{\lambda}\widetilde{R}(C_B)$. Since $C_B \in Z(U(\widetilde{L}))$, this is an L-intertwiner (Details?), and by $(**)$, $r(w) = w$ for all $w \in W$. ?

Altogether, by Lemma 5.3.4 the sequence $0 \to W \to V \to Q \to 0$ splits.

Step 2. Assume $\dim(Q) = 1$, *but* W *arbitrary finite-dimensional:* We proceed by induction on $\dim(V)$. Let thus

$$A(n) := \text{“}0 \to W \xrightarrow{\iota} V \xrightarrow{\pi} Q \to 0 \text{ with } \dim(Q) = 1 \text{ and } \dim(V) \leq n \text{ splits”}$$

$A(1)$: trivial as then $W = \{0\}$.

$A(n-1) \Rightarrow A(n)$: We may assume $W \neq \{0\}$ and $\dim(V) = n$. If W is irreducible, $A(n)$ follows from step 1. If W is not irreducible, there is a submodule $\{0\} \subsetneq S \subsetneq W$. Then

$$0 \to W/S \to V/S \to Q \to 0$$

is a short exact sequence of L-modules (Why?). This allows one to apply $A(n-1)$ and to conclude that $A(n)$ holds (Exercise.) ?

E (Ex. 27)

Step 3. Q, W arbitrary finite-dimensional: We want to construct an L-module intertwiner $r : V \to W$ which splits $0 \to W \to V \to Q \to 0$.

Recall that the space of all linear maps $\mathrm{Hom}(V, W)$ is an L-module as in Remark 2.2.6 (1). Define

$$B = \{f \in \mathrm{Hom}(V, W) \,|\, f|_W = \lambda\, \mathrm{id}_W \text{ for some } \lambda \in \mathbb{C}\} \,,$$
$$A = \{f \in \mathrm{Hom}(V, W) \,|\, f|_W = 0\} \subset B \,.$$

Claim: A and B are L-submodules of $\mathrm{Hom}(V, W)$.

(Exercise.) E (Ex. 27)

Pick any vector space projection $P : V \to W$, i.e. $P|_W = \mathrm{id}_W$. Then $B = A \oplus \mathbb{C}P$. Indeed, for $f \in B$ there is a $\lambda \in \mathbb{C}$ such that $f|_W = \lambda\, \mathrm{id}_W$, and so $f = (f - \lambda P) + \lambda P$ is a decomposition of f in $A \oplus \mathbb{C}P$. Thus B/A is one-dimensional and we can apply step 2 to the short exact sequence

$$0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} A/B \to 0 \ .$$

This shows $B = A \oplus C$ as L-modules, with C 1-dimensional. Let $f \in C$ be non-zero. Then $f|_W = \lambda \operatorname{id}_W$ (by definition as $C \subset B$), and $\lambda \neq 0$ (or else $f \in A$, contradiction).

But C is a 1-dimensional L-module, and so the L-action on C is trivial (Corollary 5.3.7). This implies that $f \in C$ is an intertwiner (Remark 2.2.6 (2)). By construction, $r = \frac{1}{\lambda} f : V \to W$ is a splitting map for the original sequence $0 \to W \to V \to Q \to 0$. □

- Let K, L be Lie algebras over a field k and let M be a K-module and U an L-module. Then $M \otimes_k U$ (tensor product of k-vector spaces) is a $K \oplus L$-module via

$$(k \oplus l).(m \otimes u) = (k.m) \otimes u + m \otimes (l.u)$$

(Why is this a $K \oplus L$-module?). ?

- If K and L are fssc Lie algebras, then so is $K \oplus L$ (by Theorem 5.1.6 (3)). We get the following corollary to Weyl's Theorem:

Corollary 5.3.8. Let W be a simple $K \oplus L$-module. Then $W \cong M \otimes U$ as $K \oplus L$-modules, where M is a simple K-module and U a simple L-module.

Proof. By Weyl's Theorem, W is fully reducible as a K-module, and we can write

$$W \cong \bigoplus_{a \in A} M_a^{\oplus n_a} \ ,$$

where the M_a are irreducible pairwise non-isomorphic K-modules, and $n_a \in \mathbb{Z}_{>0}$ denotes their multiplicity in the direct sum decomposition. Since L acts by K-intertwiners only, and there are no non-zero K-intertwiners between M_a and M_b for $a \neq b$, each $M_a^{\oplus n_a}$ is a $K \oplus L$-invariant subspace. By irreducibility of W, there can only be one: $W \cong M^{\oplus n} \cong M \otimes_{\mathbb{C}} \mathbb{C}^n$, with M a simple K-module.

Since each $l \in L$ acts as a K-intertwiner on $M \otimes_{\mathbb{C}} \mathbb{C}^n$, by Schur's Lemma it is of the form $\operatorname{id}_M \otimes_{\mathbb{C}} F(l)$ for some linear map $F(l) : \mathbb{C}^n \to \mathbb{C}^n$ (That is ? not quite Schur's Lemma - what are the missing details?). $F(-)$ turns $\mathbb{C}^n$ into an L-module, call it U, which again has to be irreducible. Else, again by Weyl's Theorem, there is a non-trivial decomposition $U = U_1 \oplus U_2$ of U, and hence also $W \cong M \otimes_{\mathbb{C}} U \cong M \otimes_{\mathbb{C}} U_1 \oplus M \otimes_{\mathbb{C}} U_1$). □

Remark 5.3.9. The above corollary implies that the simple $K \oplus L$-modules are precisely the modules isomorphic to $M \otimes_{\mathbb{C}} U$, where M is a simple K-module and U is a simple L-module. Together with Weyl's theorem, this shows that finite-dimensional representations of a fssc Lie algebra L can be understood in terms of the finite-dimensional irreducible representations of the simple summands of L.

■ For info: Let K, L be Lie algebras over a field k. Let M, N be finite-dimensional K-modules, U, V finite-dimensional L-modules. The map

$$\begin{aligned} \Phi : \mathrm{Hom}_K(M,N) \otimes_k \mathrm{Hom}_L(U,V) &\longrightarrow \mathrm{Hom}_{K\oplus L}(M \otimes_k U, N \otimes_k V) \\ f \otimes g &\longmapsto \big(m \otimes u \mapsto f(m) \otimes g(u)\big) \end{aligned}$$

is an isomorphism. To see this, first observe that for just linear maps $\mathrm{Hom}(M,N)$, etc, the above map is an isomorphism. Then check that it is an intertwiner for the $K \oplus L$ action on the hom spaces on both sides. Hence the subspaces on which $K \oplus L$ acts trivially get mapped to each other, too.

5.4 Cartan subalgebras

In this subsection L denotes a fssc Lie algebra, and we write $\kappa = \kappa_L$ for its Killing form.

Definition 5.4.1. 1. 1) A Lie subalgebra $H \subset L$ is called a *toral subalgebra* if

- H is abelian
- every $h \in H$ is ad-diagonalisable on L (i.e. $\mathrm{ad}(h) : L \to L$ is diagonalisable)

2. A toral subalgebra is maximal, if it is not properly contained in a larger toral subalgebra (i.e. $H \subset H'$ and H' toral in L implies $H = H'$).

3. A maximal toral subalgebra of a fssc Lie algebra is called a *Cartan subalgebra* (or *CSA* for short).

■ For a general finite dimensional Lie algebra, the definition of a Cartan subalgebra is different (but specialises to the above), see →EOM.

Remark 5.4.2 (For Info)**.** The reason for the name "toral" is as follows. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and $G \subset GL(n, \mathbb{K})$ a matrix Lie group, $g \subset gl(n, \mathbb{K})$ its Lie algebra, and exp : $g \to G$ the matrix exponential. If $h \subset g$ is abelian, then for all $a, b \in h$ we have $\exp(a)\exp(b) = \exp(a+b)$, i.e.

$$\exp : (g, +) \longrightarrow (G, \cdot)$$

is a group homomorphism. If $T := \exp(h) \subset G$ is compact, then it is a torus, i.e. $T \cong h/\ker(\exp|_h)$ as Lie groups. If G is compact and $h_{\mathbb{C}} \subset g_{\mathbb{C}}$ is maximal toral, then T is automatically compact (and so a torus). See Knapp, Lie groups beyond an introduction (Birkhäuser 2002), Section IV.5. (8.12.)

■ Recall from Theorem 3.3.8 the Jordan-Chevallay decomposition $x = x_s + x_n$ of a linear map $x : V \to V$ for a finite-dimensional $\mathbb{C}$-vector space V. The next proposition generalises this to fssc Lie algebras.

Theorem 5.4.3. For each $y \in L$ there are unique $y_s, y_n \in L$ such that $y = y_s + y_n$ with $[y_s, y_n] = 0$, and $\mathrm{ad}(y_s) \in \mathrm{End}(L)$ is semisimple, and $\mathrm{ad}(y_n)$ is nilpotent.

Proof. Existence: Let $y \in L$ be given and abbreviate $x = \mathrm{ad}(y) : L \to L$. Let $x = x_s + x_n$ be the decomposition into a semisimple and a nilpotent endomorphism of L as in Theorem 3.3.8. Existence follows if we can show that x_s lies in the image of ad (then there is $y_s \in L$ with $x_s = \mathrm{ad}(y_s)$ and hence also $x_n = x - x_s = \mathrm{ad}(y - y_s)$). By Proposition 5.1.8, every derivation on L is inner. Existence hence therefore follows from:

Claim: $x_s \in \mathrm{Der}(L)$.

⌈ Write L_σ for the generalised eigenspace of x of eigenvalue $\sigma \in \mathbb{C}$:

$$L_\sigma = \{v \in L \,|\, (x - \sigma\,\mathrm{id})^k v = 0 \text{ for some } k \geq 0\} .$$

Equivalently, $L_\sigma = \ker(x_s - \sigma\,\mathrm{id})$, i.e. $x_s|_{L_\sigma} = \sigma\,\mathrm{id}_{L_\sigma}$. Let now $a \in L_\sigma$, $b \in L_\tau$. Then $[a, b] \in L_{\sigma+\tau}$. To see this, first use that $x = \mathrm{ad}(y) \in \mathrm{Der}(L)$ to show, for $n \geq 0$,

$$\big(x - (\sigma+\tau)\,\mathrm{id}\big)^n [a, b] = \sum_{k=0}^{n} \binom{n}{k} \big[\,(x - \sigma\,\mathrm{id})^k a\,,\,(x - \tau\,\mathrm{id})^{n-k} b\,\big] .$$

(Why does this hold? *Hint:* Check that $\big(x - (\sigma+\tau)\,\mathrm{id}\big)[a, b] = [xa, b] + [a, xb] - [\sigma a, b] - [a, \tau b]$ and use induction.) For n large enough, for each k one of the two terms in the Lie bracket on the right hand side is zero, showing that indeed $[a, b] \in L_{\sigma+\tau}$.

To show that x_s is a derivation, we need to check that for all $a, b \in L$ we have

$$x_s([a,b]) = [x_s(a), b] + [a, x_s(b)] .$$

Since L is a direct sum of the generalised eigenspaces L_σ, it is enough to check the above identity for all $a \in L_\sigma$, $b \in L_\tau$ ($\sigma, \tau \in \mathbb{C}$). But in that case it is an immediate from $[a,b] \in L_{\sigma+\tau}$. ⌋

Uniqueness: This follows from uniqueness of $x = x_s + x_n$ in Theorem 3.3.8, together with Lemma 3.3.10 and Remark 5.1.7. (Details?) □ (?)

Lemma 5.4.4. A non-zero L has a non-zero CSA.

Proof. Given a toral subalgebra of L, we can always extend it to a maximal one (Why?). It remains to show that a non-zero toral subalgebra exists. For this, it is enough to show that there is a $y \in L$ such that in the decomposition $y = y_s + y_n$ from Theorem 5.4.3 we have $y_s \neq 0$ (for then $\mathbb{C}y_s$ is a toral subalgebra). (?)

Suppose that for each $y \in L$ we have $y_s = 0$, i.e. $y = y_n$. By definition, this means that each $y \in L$ is ad-nilpotent. Then Engel's Theorem (Theorem 3.1.3) implies that L is nilpotent, and so in particular solvable (Remark 3.2.2). But then $\mathrm{Rad}(L) = L \neq \{0\}$, in contradiction to semi-simplicity of L (Definition 3.2.6). □

■ Let $H \subset L$ be a CSA. Define, for $\alpha \in H^*$,

$$L_\alpha := \{x \in L \,|\, \mathrm{ad}(h)x = \alpha(h)x\} ,$$

i.e. L_α is the common eigenspace for the eigenvalue functional α.

Definition 5.4.5. A *root of L relative to H* is an $\alpha \in H^*$ such that $\alpha \neq 0$ and $L_\alpha \neq \{0\}$. The set of roots is called the *root system (of L relative to H)* and is denoted by

$$\Phi = \Phi(L, H) = \{\alpha \in H^* \,|\, \alpha \text{ is a root}\} .$$

Remark 5.4.6. By definition

$$L_0 = \{x \in L \,|\, \mathrm{ad}(h)x = 0\} .$$

This subspace is also called the *centraliser* $C_L(H)$ of the abelian Lie algebra H in L. We have the direct sum decomposition (as a vector space, not as a Lie algebra)

$$L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha .$$

This is called the *root space decomposition of L (relative to H)*.

(Why does one obtain all of L, and why does $x_0 + \sum_{\alpha\in\Phi} x_\alpha = 0$ with $x_0 \in L_0$, $x_\alpha \in L_\alpha$ imply that x_0 and all x_α are zero? *Hint:* Recall the argument for the corresponding property of eigenspaces for different eigenvalues and generalise to the present situation with eigenvalue functionals.) ?

Proposition 5.4.7. Let $H \subset L$ be a CSA. Then

1. $\kappa|_{H\times H}$ is non-degenerate,
2. $H = C_L(H)$,
3. $\mathrm{span}_{\mathbb{C}}\Phi(L,H) = H^*$.

The proof of this proposition relies on the next lemma.

Lemma 5.4.8. Let $H \subset L$ be a CSA. Then:

1. For all $\alpha, \beta \in H^*$: $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$.
2. Let $x \in L_\alpha$, $y \in L_\beta$. If $\alpha + \beta \neq 0$, then $\kappa(x,y) = 0$.
3. $\kappa|_{L_0\times L_0}$ is non-degenerate.
4. Let $n, x \in L$ with n is ad-nilpotent and $[n,x] = 0$. Then $\kappa(n,x) = 0$.

Proof. Part 1: For all $x \in L_\alpha$, $y \in L_\beta$, $h \in H$ we have

$$\begin{aligned}\mathrm{ad}(h)[x,y] &\overset{(*)}{=} [\mathrm{ad}(h)x, y] + [x, \mathrm{ad}(h)y] = [\alpha(h)x, y] + [x, \beta(h)y] \\ &= (\alpha+\beta)(h)\,[x,y]\end{aligned}$$

where $(*)$ is the statement that $\mathrm{ad}(h)$ is a deriviation on L (Example 2.2.3 (2)). The above computation shows $[x,y] \in L_{\alpha+\beta}$, and so $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$.

Part 2: Let $x \in L_\alpha$, $y \in L_\beta$, $h \in H$. Then on the one hand

$$\kappa(\mathrm{ad}(h)x, y) = \kappa(\alpha(h)x, y) = \alpha(h)\,\kappa(x,y) \ ,$$

and on the other hand

$$\kappa(\mathrm{ad}(h)x, y) = \kappa(-[x,h], y) = -\kappa(x, [h,y]) = -\beta(h)\,\kappa(x,y) \ .$$

It follows that for all $h \in H$, $(\alpha+\beta)(h)\,\kappa(x,y) = 0$. If $\alpha+\beta \neq 0$, there is an $h \in H$ such that $(\alpha+\beta)(h) \neq 0$, and hence we must have $\kappa(x,y) = 0$.

Part 3: Let $x \in L_0$, $x \neq 0$. As κ is non-degenerate on all of L, there is a $y \in L$ such that $\kappa(x,y) \neq 0$. Write y in terms of its components in the root space decomposition (Remark 5.4.6), $y = y_0 + \sum_{\alpha\in\Phi} y_\alpha$, where $y_0 \in L_0$ and $y_\alpha \in L_\alpha$. Then

$$0 \neq \kappa(x,y) = \kappa(x,y_0) + \sum_{\alpha\in\Phi} \kappa(x,y_\alpha) \overset{(*)}{=} \kappa(x,y_0)\ ,$$

where in $(*)$ we used that by part 2 $\kappa(x,y_\alpha)=0$ (as $x\in L_0$, but $\alpha\neq 0$ as α is a root). Thus we can find $y_0 \in L_0$ such that $\kappa(x,y_0)\neq 0$, and so κ remains non-degenerate when restricted to L_0.

Part 4: The composition $\mathrm{ad}(n)\circ\mathrm{ad}(y) : L \to L$ is nilpotent (Why?). A nilpotent endomorphism has trace zero. ? □

Proof of Proposition 5.4.7. Part 1:

Claim 1: Let $y \in L_0$ and write $y = y_s + y_n$ as in Theorem 5.4.3. Then $y_s \in H \subset L_0$ and $y_n \in L_0$.

⌈ We first show that $y_s, y_n \in H$. That is, we need to show that for all $h \in H$ we have $[h,y_s] = 0 = [h,y_n]$, or, equivalently, that $\mathrm{ad}(y_s)h = 0 = \mathrm{ad}(y_n)h$. By Theorem 3.3.8 (and Lemma 3.3.10) there is a polynomial p with $p(0)=0$ such that $\mathrm{ad}(y_s) = p(\mathrm{ad}(y))$. Since $y\in L_0$ we have $\mathrm{ad}(y)h = 0$, and hence also $\mathrm{ad}(y_s)h=0$. That $\mathrm{ad}(y_n)h=0$ follows analogously.

Finally, y_s is ad-diagonalisable by definition, and $[y_s,H]=\{0\}$ since $y_s\in L_0$. Hence $H+\mathbb{C}y_s$ is again a toral subalgebra. Maximality of H implies $y_s \in H$. ⌋

Let $h\in H$, $h\neq 0$ be given. By Lemma 5.4.8 (3), the Killing form is non-degenerate on L_0, and so there is a $y \in L_0$ such that $\kappa(h,y)\neq 0$. Writing $y = y_s+y_n$ we get $\kappa(h,y_s)+\kappa(h,y_n)\neq 0$. By Claim 1, $y_n\in L_0$, and so $[h,y_n]=0$. By Lemma 5.4.8 (4) we see $\kappa(h,y_n)=0$. Thus $\kappa(h,y_s)\neq 0$, and since by Claim 1 also $y_s\in H$, we conclude that κ is non-degenerate when restricted to H.

Part 2: By Lemma 5.4.8 (1), L_0 is a Lie-subalgebra of L. We want to show that $L_0 = H$.

Claim 2: Every $y \in L_0$ is ad-nilpotent on L_0.

⌈ Decompose $y = y_s + y_n$ in L. By Claim 1, $y_s\in H$, and so by definition of L_0 we have $\mathrm{ad}(y_s)x = 0$ for all $x\in L_0$. In other words, $y_s|_{L_0}=0$. But then $y|_{L_0} = y_n|_{L_0}$, and so y is ad-nilpotent on L_0. ⌋

Claim 3: L_0 is abelian.

⌈ Claim 2 and Engel's Theorem (Theorem 3.1.3) show that L_0 is a nilpotent Lie algebra, and so in particular solvable. Let $y \in [L_0, L_0]$ be given. Then for each $x \in L_0$ we have, by Corollary 3.3.5 applied to the Lie algebra L_0, that $\kappa(x, y) = 0$. But κ is non-degenerate on L_0 (Lemma 5.4.8 (3)), and so we must have $y = 0$. This shows that $[L_0, L_0] = \{0\}$. ⌋

Let $y \in L_0$ be given, and decompose it as $y = y_s + y_n$. Since by Claim 1, $y_s \in H$, it remains to show that $y_n = 0$. For any $x \in L_0$ we have $[x, y_n] = 0$ as L_0 is abelian, and so $\kappa(x, y_n) = 0$ by Lemma 5.4.8 (4). Using once more non-degeneracy of κ on L_0, we conclude that $y_n = 0$.

Part 3: Write $F = \mathrm{span}_{\mathbb{C}}\Phi(L, H)$. Suppose $F \subsetneq H^*$. Then there is a non-zero $h \in H$ such that $\varphi(h) = 0$ for all $\varphi \in F$ (Why?). We will now show that then $\mathrm{ad}(h) : L \to L$ is zero, in contradiction to non-degeneracy of κ. (What does $\mathrm{ad}(h) = 0$ have to do with non-degeneracy?) ? ?

Recall the root space decomposition $L = L_0 \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$ from Remark 5.4.6. For $x \in L_0$ we have $\mathrm{ad}(h)x = 0$ as $h \in H = L_0$ (by part 2) and H is abelian. For $x \in L_\alpha$ ($\alpha \in \Phi$), we get $\mathrm{ad}(h)x = \alpha(h)x = 0$ as well. Hence $\mathrm{ad}(h) = 0$ on all of L. □ (12.12.)

Corollary 5.4.9. Let $H \subset L$ be a toral subalgebra. Then H is a CSA iff $C_L(H) = H$.

Proof. $\Rightarrow$: Proposition 5.4.7 (2). $\Leftarrow$: Suppose H' is a toral subalgebra such that $H \subset H'$. Let $x \in H'$. As H' is abelian, we in particular have $\mathrm{ad}(h)x = 0$ for all $h \in H$. Thus $x \in C_L(H)$. Altogether $H \subset H' \subset C_L(H)$. □

Example 5.4.10. $L = sl(n, \mathbb{C})$:
Set

$$H = \{ D \in sl(n, \mathbb{C}) \,|\, D \text{ is a diagonal matrix } \} .$$

Thus H consists of traceless diagonal $n \times n$ matrices and hence $\dim(H) = n - 1$.

- *H is a toral subalgebra:* H is clearly abelian. It remains to show that each $h \in H$ is ad-diagonalisable. In fact, we can diagonalise it explicitly on $gl(n, \mathbb{C})$. Recall the unit matrices $E_{ij} \in \mathrm{Mat}(n, \mathbb{C})$ from Section 1.6. Write $h = \sum_{k=1}^n \lambda_k E_{kk}$. Since h is trace-less we have $\sum_k \lambda_k = 0$. We have, for all $i, j = 1, \dots, n$,

$$\mathrm{ad}(h)E_{ij} = \sum_{k=1}^n \lambda_k \big(E_{kk}E_{ij} - E_{ij}E_{kk}\big) = \sum_{k=1}^n \lambda_k \big(\delta_{k,i}E_{kj} - \delta_{j,k}E_{ik}\big)$$

$$= (\lambda_i - \lambda_j)E_{ij} \ .$$

In particular, $\mathrm{ad}(h)$ acts diagonally on the basis $\{E_{kk} - E_{nn} | k = 1, \dots, n-1\} \cup \{E_{ij} | i \neq j\}$ of $sl(n, \mathbb{C})$.

- *H is a CSA:* By Corollary 5.4.9 it is enough to show that $C_L(H) = H$. We need to find those $M = \sum_{i,j=1}^n \mu_{ij} E_{ij} \in sl(n, \mathbb{C})$ such that for all $h = \sum_{k=1}^n \lambda_k E_{kk} \in H$ we have $[h, M] = 0$. We compute

$$[h, M] = \sum_{i,j}^n \mu_{ij}[h, E_{ij}] = \sum_{i,j}^n \mu_{ij}(\lambda_i - \lambda_j)E_{ij} \ .$$

  Since the λ_i can be chosen arbitrarily (subject to $\sum_k \lambda_k = 0$), and the E_{ij} are linearly independent, this can only be zero for all choices of h if $\mu_{ij} = 0$ for $i \neq j$, i.e. for diagonal M. This shows the inclusion $C_L(H) \subset H$. And since anyway $H \subset C_L(H)$ we get equality.

- *Root system Φ:* From the explicit diagonalisation of $\mathrm{ad}(h)$ above, we read off the roots (i.e. the eigenvalue functions $\alpha \in H^*$ which are not zero and for which L_α is not zero. Namely, we set $\alpha_{ij}(h) = \lambda_i - \lambda_j$ for $h = \sum_{k=1}^n \lambda_k E_{kk}$. Then

$$\Phi = \{\alpha_{ij} \,|\, i, j = 1, \dots, n \text{ and } i \neq j\} \ ,$$

  i.e. $sl(n, \mathbb{C})$ has $n^2 - n$ roots. The corresponding eigenspaces are

$$\begin{aligned} L_0 = H &= \mathrm{span}_{\mathbb{C}}\{E_{kk} - E_{nn} | k = 1, \dots, n-1\} \ , \\ L_{\alpha_{ij}} &= \mathbb{C}E_{ij} \qquad \text{for } i \neq j \ . \end{aligned}$$

As a consistency check, let us compute the dimension of $sl(n, \mathbb{C})$ via its root space decompostion,

$$sl(n, \mathbb{C}) = L_0 \oplus \bigoplus_{i \neq j} L_{\alpha_{ij}} \ .$$

Here, $\dim L_0 = n - 1$ and $\dim L_{\alpha_{ij}} = 1$ with a total of $n^2 - n$ roots, giving $\dim(sl(n, \mathbb{C})) = n-1+n^2-n = n^2-1$, which is of course the correct answer.

5.5 Root space decomposition

In this subsection we fix a fssc Lie algebra L and a CSA $H \subset L$. We write κ for the Killing form of L and Φ for its root system relative to H. By Proposition 5.4.7, the root space decomposition of L is

$$L \ = \ H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha \ .$$

■ For $\varphi \in H^*$ we define $H^\varphi \in H$ to be the unique element such that

$$\varphi = \kappa(H^\varphi, -) .$$

Alternatively, define $\tilde{\kappa} : H \to H^*$, $a \mapsto \kappa(a, -)$. Since $\kappa|_{H \times H}$ is non-degenerate (Proposition 5.4.7), this is an isomorphism. Then

$$H^\varphi = \tilde{\kappa}^{-1}(\varphi) .$$

Define a bilinear pairing on H^* via

$$(-,-) : H^* \times H^* \to \mathbb{C} \quad , \quad (\varphi, \psi) := \kappa(H^\varphi, H^\psi) .$$

This pairing is non-degenerate (Why?) and by definition we have ?

$$(\varphi, \psi) = \varphi(H^\psi) = \psi(H^\varphi) .$$

Lemma 5.5.1. Let $\alpha \in \Phi$.

1. $-\alpha \in \Phi$.
2. For $x \in L_\alpha$, $y \in L_{-\alpha}$ we have $[x, y] = \kappa(x, y)H^\alpha$.
3. $(\alpha, \alpha) \neq 0$.

Proof. Part 1: By Lemma 5.4.8 (2) and non-degeneracy of κ, if $L_\alpha \neq \{0\}$ also $L_{-\alpha}$ must be non-zero.

Part 2: By Lemma 5.4.8 (1) and Proposition 5.4.7 (2), $[x, y] \in L_0 = H$. Since also $H^\alpha \in H$ and $\kappa|_{H \times H}$ is non-degenerate (Proposition 5.4.7 (1)), it is enough to check that for all $k \in H$ we have $\kappa(k, [x, y]) = \kappa(k, \kappa(x, y)H^\alpha$. Indeed,

$$\kappa(k, [x, y]) = \kappa([k, x], y) = \kappa(\alpha(k)x, y) = \kappa(x, y)\,\alpha(k) \ ,$$
$$\kappa(k, \kappa(x, y)H^\alpha) = \kappa(x, y)\kappa(H^\alpha, k) = \kappa(x, y)\,\alpha(k) \ ,$$

where in the first line we used $x \in L_\alpha$ and in the second line that $\alpha = \kappa(H^\alpha, -)$.

Part 3: Suppose $(\alpha, \alpha) = 0$. Choose $x \in L_\alpha$, $x \neq 0$, and $y \in L_{-\alpha}$ such that $\kappa(x, y) \neq 0$ (possible by Lemma 5.4.8 (2) and non-degeneracy of κ). Define

$$A = \mathrm{span}_{\mathbb{C}}\{x, y, H^\alpha\} \subset L .$$

The non-trivial Lie brackets are:

$$[x, y] = \kappa(x, y)H^\alpha \qquad \text{(by part 2)} \ ,$$

$$[H^\alpha, x] = \alpha(H^\alpha)x = (\alpha,\alpha)x = 0 \qquad \text{(as } x \in L_\alpha) ,$$
$$[H^\alpha, y] = (-\alpha)(H^\alpha)y = -(\alpha,\alpha)y = 0 \qquad \text{(as } y \in L_{-\alpha}) .$$

In particular, A is a Lie subalgebra with $[A, A] \subset \mathbb{C}H^\alpha$, and $[A, [A, A]] = \{0\}$, so that A is solvable. Set $\widetilde{A} := \mathrm{ad}(A) \subset gl(L)$. By Proposition 3.2.4 (1) also $\widetilde{A}$ is solvable. By Lie's Theorem (Theorem 3.2.3), $\widetilde{A}$ stabilises a flag in L. In terms of a basis adapted to that flag, this means that for each $f \in \widetilde{A}$, the representing matrix of f is upper triangular (sec:nilpotent-and-Engel). Since the commutator of two upper triangular matrices is strictly upper triangular (see the main example in Section 3.1), we have that

$$\mathrm{ad}(H^\alpha) = \frac{1}{\kappa(x,y)} \big[\, \mathrm{ad}(x), \mathrm{ad}(y)\big]$$

is represented by a strictly upper triangular matrix. In particular $\mathrm{ad}(H^\alpha) : L \to L$ is nilpotent. (Why is it not enough to know that $[A, [A, A]] = \{0\}$? Does this not mean that every element of A is ad-nilpotent?) On the other had, buy definition $\mathrm{ad}(H^\alpha)$ is diagonalisable, and an linear map that is diagonalisable (i.e. semisimple) and nilpotent is zero (cf. the comment below Defintion 3.3.7): $\mathrm{ad}(H^\alpha) = 0$. But then $\kappa(H^\alpha, -) = 0$, in contradiction to non-degeneracy of the Killing form. ?

Thus $(\alpha, \alpha) \neq 0$. □

■ For $\alpha \in \Phi$ define

$$h_\alpha := \frac{2}{(\alpha,\alpha)} H^\alpha .$$

Proposition 5.5.2. Let $\alpha \in \Phi$, and let $e_\alpha \in L_\alpha$, $f_\alpha \in L_{-\alpha}$ satisfy $\kappa(e_\alpha, f_\alpha) = 2/(\alpha,\alpha)$. Then the linear map

$$\varphi_\alpha : sl(2,\mathbb{C}) \to L \quad , \quad e \mapsto e_\alpha \ , \quad h \mapsto h_\alpha \ , \quad f \mapsto f_\alpha$$

is a Lie algebra homomorphism.

Proof. This follows from computing the following Lie brackets (Why?): ?

$$[e_\alpha, f_\alpha] = \kappa(e_\alpha, f_\alpha)\, H^\alpha = h_\alpha \qquad \text{(by Lemma 5.5.1 (2))} ,$$
$$[h_\alpha, e_\alpha] = \alpha(h_\alpha)\, e_\alpha = \frac{2}{(\alpha,\alpha)} \alpha(H^\alpha)\, e_\alpha = 2e_\alpha ,$$
$$[h_\alpha, f_\alpha] = (-\alpha)(h_\alpha) = -2f_\alpha .$$

□

■ Let $\alpha \in \Phi$. Write $sl_\alpha \subset L$ for the Lie subalgebra given by the image of $sl(2,\mathbb{C})$ under φ_α. By definition, sl_α is isomorphic to $sl(2,\mathbb{C})$ as a Lie algebra.

Remark 5.5.3. Let M be a finite-dimensional $sl(2,\mathbb{C})$-module. We will need the following properties of such modules:

1. *Claim:* The action of h on M is diagonalisable, and its eigenvalues are integers.

   (Why? *Hint:* Use Weyl's Theorem to reduce the question to irreducible representations, and Exercise 25 to read off the possible h-eigenvalues for finite-dimensional irreducible $sl(2,\mathbb{C})$-modules. And: Can you give a counter-example to the claim if one drops the condition that M should be finite-dimensional?) ?

2. For $\lambda \in \mathbb{C}$ write

   $$M[\lambda] := \{u \in M \mid h.u = \lambda u\} \ .$$

   Claim: The number n of irreducible summands in M is given by $n = \dim M[0] + \dim M[1]$.

   (Exercise). E (Ex. 31)

   In other words, $M = \bigoplus_{k=1}^{n} N_k$, where each N_k is an irreducible $sl(2,\mathbb{C})$-module.

3. *Claim:* If $M[\lambda] \neq 0$, then also $M[\mu] \neq 0$ for $\mu \in \{-|\lambda|, -|\lambda| + 2, \dots, |\lambda| - 2, |\lambda|\}$ (steps of 2).

   (Why? *Hint:* Use the same hint as in point 1.) ?

Lemma 5.5.4. Let $\alpha \in \Phi$. Then

$$\mathbb{C}H^\alpha \oplus \bigoplus_{k\in\mathbb{Z},k\neq 0} L_{k\alpha}$$

is an irreducible sl_α-module.

Proof. Write M for the direct sum in the statement.

M is an sl_α-module: It is enough to check that for any $x \in \mathbb{C}H^\alpha$ or in $L_{k\alpha}$ we have

$$[e_\alpha, x] \, , \ [f_\alpha, x] \, , \ [h_\alpha, x] \ \in \ M \ . \tag{$*$}$$

Recall from Lemma 5.4.8 (1) that for any $\lambda, \mu \in H^*$ we have $[L_\lambda, L_\mu] \subset L_{\lambda+\mu}$. Since $e_\alpha \in L_\alpha$, $h_\alpha \in L_0$, $f_\alpha \in L_{-\alpha}$, this implies ($*$) unless $k \in \{\pm 1\}$ (Details?). For $k \in \{\pm 1\}$ it only remains to compute (Why is this enough?) ?

$$k = 1 : \quad [f_\alpha, x] = -[x, f_\alpha] = \kappa(x, f_\alpha) H^\alpha \ ,$$
$$k = -1 : \quad [e_\alpha, x] = \kappa(e_\alpha, x) H^\alpha \ ,$$

both of which lie in M, and where in both cases we used Lemma 5.5.1 (2).

M is irreducible: By Remark 5.5.3 (2) we need to check that $\dim M[0] + \dim M[1] = 1$. The direct sum decomposition of M in the statement of the lemma is already the decomposition into $\mathrm{ad}(h^\alpha)$-eigenspaces. For $x \in L_{k\alpha}$ for any $k \in \mathbb{Z}$, we compute

$$[h_\alpha, x] = (k\alpha)(h^\alpha)\, x = \frac{2}{(\alpha,\alpha)} k\alpha(H^\alpha)\, x = 2k\, x$$

Thus $M[0] = \mathbb{C}H^\alpha$, $M[1] = \{0\}$, and so $\dim M[0] = 1$, $\dim M[1] = 0$. $\square$ (15.12.)

Theorem 5.5.5. Let $\alpha, \beta \in \Phi$. Then:

1. If $\lambda\, \alpha \in \Phi$ for some $\lambda \in \mathbb{C}$, then $\lambda \in \{\pm 1\}$.
2. $\dim L_\alpha = 1$.
3. $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.
4. If $\beta \neq \pm\alpha$, then $\bigoplus_{k\in\mathbb{Z}} L_{\beta+k\alpha}$ is an irreducible sl_α-module.
5. If $\alpha + \beta \in \Phi$, then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.

Proof. We will first show part 1 under the additional assumption that $\lambda \in \mathbb{Z}$, then we show parts 2,3, then the general case of part 1 (i.e. $\lambda \in \mathbb{C}$), and then parts 4,5.

Part 1a and part 2: Consider the irreducible sl_α-module $M = \mathbb{C}H^\alpha \oplus \bigoplus_{k\in\mathbb{Z},k\neq 0} L_{k\alpha}$ from Lemma 5.5.4. By its definition below Proposition 5.5.2,

$$sl_\alpha \ = \ \mathbb{C}f_\alpha \oplus \mathbb{C}H^\alpha \oplus \mathbb{C}e_\alpha \qquad , \quad \text{where} \quad f_\alpha \in L_{-\alpha}, e_\alpha \in L_\alpha \ .$$

In particular, $sl_\alpha \subset M$. But at the same time, sl_α is an sl_α-submodule of L (Why?), and as M is irreducible, we must have $M = sl_\alpha$. ?

It follows that $L_{k\alpha} = \{0\}$ for $|k| \geq 2$, showing part 1 of the theorem for $\lambda \in \mathbb{Z}$. It also follows that $L_\alpha = \mathbb{C}e_\alpha$ and $L_{-\alpha} = \mathbb{C}f_\alpha$, so that both are one-dimensional, showing part 2.

Part 3: By Remark 5.5.3 (1), the eigenvalues of the linear map $\mathrm{ad}(h_\alpha) : L \to L$ are integers. Let $x \in L_\beta$ be non-zero. Then

$$\mathrm{ad}(h_\alpha)\, x = \frac{2}{(\alpha,\alpha)}[H^\alpha, x] = \frac{2}{(\alpha,\alpha)}\beta(H^\alpha)\, x = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\, x \ .$$

Thus x is an eigenvector of h_α of eigenvalue $2(\alpha, \beta)/(\alpha, \alpha)$, which hence must be an integer.

Part 1b: Suppose $\lambda\alpha \in \Phi$ for some $\lambda \in \mathbb{C}$. By part 3,

$$\frac{2(\alpha, \lambda\alpha)}{(\alpha, \alpha)} = 2\lambda \in \mathbb{Z} \ .$$

Thus $\lambda = m/2$ for some $m \in \mathbb{Z}$. Write $\gamma := \lambda\alpha$ Then $\gamma \in \Phi$ and also $\lambda^{-1}\gamma \in \Phi$ and the above reasoning gives $2\lambda^{-1} = \frac{4}{m} \in \mathbb{Z}$. The remaining possibilities are $m \in \{\pm 1, \pm 2, \pm 4\}$. It remains to exclude the cases $m = \pm 1$ and $m = \pm 4$.

- Suppose $m = \pm 4$: Then $\lambda = \pm 2 \in \mathbb{Z}$ and this case was already excluded in part 1a.
- Suppose $m = \pm 1$: Then $\lambda^{-1} = \pm 2 \in \mathbb{Z}$ and this is also excluded by applying part 1a to the root γ.

Part 4: As in the proof of Lemma 5.5.4 one checks that $M := \bigoplus_{k \in \mathbb{Z}} L_{\beta + k\alpha}$ is an sl_α-module (Details?). By parts 1a and 1b, $\beta + k\alpha \neq 0$ for all $k \in \mathbb{Z}$ (else $\beta = -k\alpha$, which would require $k = \pm 1$, but $\beta \neq \pm\alpha$ by assumption in the statement of part 4. ?

As $L_\beta \subset M$, we have $M \neq \{0\}$ and so it has to contain at least one irreducible summand, and so by Remark 5.5.3 (2) we have

$$\dim M[0] + \dim M[1] \ \geq \ 1 \ .$$

The h_α-action on $x \in L_{\beta+k\alpha}$ is

$$h_\alpha.x = [h_\alpha, x] = \frac{2}{(\alpha, \alpha)}\big((\alpha, \beta) + k(\alpha, \alpha)\big) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} + 2k \ .$$

Thus the differences of eigenvalues are even and hence at most one of $M[0]$ and $M[1]$ can be non-zero. As the eigenspaces $L_{\beta+k\alpha}$ are at most 1-dimensional by part 2, we get the estimate $\dim M[0] + \dim M[1] \ \leq \ 1$. Thus the sum of dimensions is equal to 1 and by Remark 5.5.3 (2), M is irreducible.

Part 5: If $\alpha+\beta \in \Phi$, then $\beta \neq \pm\alpha$ (part 1). By part 4 the module M as above is irreducible. By the explicit form of irreducible $sl(2)$-modules, if $L_\beta \neq \{0\}$ and $L_{\alpha+\beta} \neq \{0\}$, then for $x \in L_\beta$, $x \neq 0$ also $e_\alpha.x \neq 0$. Hence $[L_\alpha, L_\beta] \neq \{0\}$. But $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$ and by part 2 the latter space is 1-dimensional, we have $[L_\alpha, L_\beta] = L_{\alpha+\beta}$. $\square$

■ Let $E(L,H) \subset H^*$ (or E for short) be the real vector space given by the real span of all roots,

$$E \equiv E(L,H) := \mathrm{span}_{\mathbb{R}}\Phi \ .$$

Part 1 of the next theorem shows that E is a euclidean space (i.e. a real vector space with an inner product).

Theorem 5.5.6. 1. The bilinear form $(-,-)$ on H^*, restricted to E, is real valued and positive definite.

2. $H^* = E \oplus iE$, and in particular $\dim_{\mathbb{R}} E = \dim_{\mathbb{C}} H$.

Proof. *Part 1:* We have, for all $\lambda, \mu \in H^*$ that (E) (Ex. 35)

$$(\lambda,\mu) \overset{\text{Def.}}{=} \kappa(H^\lambda, H^\mu) \overset{\text{Ex.}}{=} \sum_{\gamma\in\Phi} \gamma(H^\lambda)\gamma(H^\mu) \overset{\text{Def.}}{=} \sum_{\gamma\in\Phi}(\gamma,\lambda)(\gamma,\mu) \ . \qquad (*)$$

Setting $\lambda = \mu = \alpha \in \Phi$ and multiplying both sides by $4/(\alpha,\alpha)^2$ gives

$$\frac{4}{(\alpha,\alpha)} = \sum_{\gamma\in\Phi}\left(\frac{2(\gamma,\alpha)}{(\alpha,\alpha)}\right)^2$$

The right hand side is an integer as by Theorem 5.5.5 (3) each summand is an integer. But then (α,α) is rational and so in particular real. Furthermore, since for all $\alpha,\beta \in \Phi$, $2(\alpha,\beta)/(\alpha,\alpha)$ is an integer, also (α,β) is rational and hence real. By definition, Φ spans E over $\mathbb{R}$, and so $(\alpha,\beta) \in \mathbb{R}$ for all $\alpha,\beta \in \Phi$ implies that $(\lambda,\mu) \in \mathbb{R}$ for all $\lambda,\mu \in E$.

To see that $(-,-)$ is positive definite, we compute

$$(\lambda,\lambda) \overset{(*)}{=} \sum_{\lambda\in\Phi}(\gamma,\lambda)^2 \ .$$

As $(\gamma,\lambda) \in \mathbb{R}$, the right hand side is ≥ 0. Thus $(\lambda,\lambda) \geq 0$ for all $\lambda \in E$. Suppose now that $(\lambda,\lambda) = 0$. Then necessarily $(\gamma,\lambda) = 0$ for all $\gamma \in \Phi$. Since Φ spans H^* over $\mathbb{C}$ and $(-,-)$ is non-degenerate on H^* (see the beginning of Section 5.5), this implies $\lambda = 0$.

Part 2: Clearly $\mathrm{span}_{\mathbb{C}}E = \mathrm{span}_{\mathbb{C}}\Phi = H^*$, and so $E + iE = H^*$. It remains to show that $E \cap iE = \{0\}$. As $(-,-)$ is positive definite on E, it must be negative definite on iE. Thus, $\lambda \in E \cap iE$ satisfies $(\lambda,\lambda) \geq 0$ and $(\lambda,\lambda) \leq 0$. Hence $(\lambda,\lambda) = 0$ and by positive definiteness on E (part 1) it follows that $\lambda = 0$. □

Example 5.5.7. *The space E for $sl(n,\mathbb{C})$:*
For $L = sl(n,\mathbb{C})$ we may choose $H = \{\text{diagonal matrices of trace } 0\}$, see Example 5.4.10. Define $\varepsilon_i \in H^*$, $i = 1,\dots,n$ via

$$\varepsilon_i(h) = h_i \quad \text{for} \quad h = \sum_{j=1}^{n} h_j E_{jj} \ .$$

Since for $h \in H$ we have $\sum_j d_j = 0$, the ε_i are not linearly independent but satisfy $\sum_{i=1}^{n} \varepsilon_i = 0$. Still, $\lambda \in H^*$ can be *uniquely* written as (Why?) ?

$$\lambda = \sum_{j=1}^{n} \lambda_j \varepsilon_j \quad , \quad \text{where} \quad \lambda_j \in \mathbb{C} \quad \text{such that} \quad \sum_{j=1}^{n} \lambda_j = 0 \ .$$

Claim: For $\lambda, \mu \in H^*$, expanded as above, we have $H^\lambda = \frac{1}{2n} \sum_{j=1}^{n} \lambda_j E_{jj}$ and

$$(\lambda, \mu) = \frac{1}{2n} \sum_{j=1}^{n} \lambda_j \mu_j \ .$$

⌈ For H^λ we have to check that for all $h \in H$ we have $\lambda(h) = \kappa(H^\lambda, h)$. The Killing form of $sl(n,\mathbb{C})$ restricted to H is given by (Exercise) E (Ex. 26)

$$\kappa(h,k) = 2n \sum_{j=1}^{n} h_j k_j \qquad \text{for} \quad h,k \in H \ .$$

Write $h = \sum_{j=1}^{n} h_j E_{jj}$. Then

$$\lambda(h) = \sum_{j=1}^{n} \lambda_j \varepsilon_j(h) = \sum_{j=1}^{n} \lambda_j h_j \ ,$$

$$\kappa(H^\lambda, h) = \frac{1}{2n} \kappa\Big(\textstyle\sum_{j=1}^{n} \lambda_j E_{jj} \, , \, h \Big) = \displaystyle\sum_{j=1}^{n} \lambda_j h_j \ .$$

Using this, we get

$$(\lambda, \mu) = \mu(H^\lambda) = \sum_{j=1}^{n} \mu_j \varepsilon_j(H^\lambda) = \frac{1}{2n} \sum_{j=1}^{n} \mu_j \lambda_j \ .$$

⌋

In the present notation, the roots of $sl(n,\mathbb{C})$ from Example 5.4.10 can be written as

$$\Phi = \{\alpha_{ij} := \varepsilon_i - \varepsilon_j \,|\, i,j = 1,\dots,n \ , \ i \neq j\} \ .$$

Then

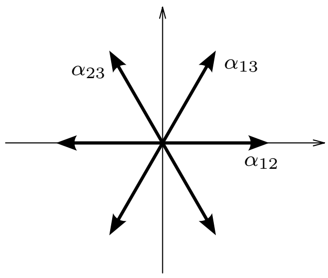
$$E = \mathrm{span}_{\mathbb{R}}\Phi = \{\sum_{j=1}^{n} s_j\varepsilon_j \,|\, s_j \in \mathbb{R}\} \quad , \qquad \dim_{\mathbb{R}} E = n-1 \ .$$

(Why can one use the ε_i directly, rather than the α_{ij}?) Some inner products are ?

$$\begin{aligned}
(\alpha_{ij}, \alpha_{ij}) &= \frac{1}{2n}(1+1) = \frac{1}{n} \ , \\
(\alpha_{ij}, \alpha_{kl}) &= 0 \text{ for } i,j,k,l \text{ pairwise distinct} \quad , \\
(\alpha_{i,i+1}, \alpha_{i+1,i+2}) &= -\frac{1}{2n} \ .
\end{aligned}$$

A d-dimensional euclidean space can be isometrically mapped to $\mathbb{R}^d$. Thus there is a linear isometry $f : E \cong \mathbb{R}^{n-1}$. We can always arrange $f(\alpha_{12}) = \frac{1}{\sqrt{n}}\, e_1$.

For example, for $sl(3,\mathbb{C})$ one obtains, with an appropriate choice of f,



where each arrow has length $1/\sqrt{3}$ and the angle between the arrows is $60°$. (Why is that the result? Can you check the various inner products?) ?

(19.12.)

Lemma 5.5.8. If $\alpha, \beta \in \Phi$ and $(\alpha,\beta) > 0$, then also $\beta - \alpha \in \Phi$ and $\alpha - \beta \in \Phi$.

Proof. We will show $\beta - \alpha \in \Phi$. That $\alpha - \beta \in \Phi$ then follows from Lemma 5.5.1 (1).

We saw in the proof of part 4 of Theorem 5.5.5 that h_α acts on $L_{\beta+k\alpha}$ by multiplication with $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2k$. By assumption $L_\beta \neq \{0\}$ and so $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} > 0$ is an eigenvalue of h_α. By Remark 5.5.3 (3), also $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} - 2$ is an eigenvalue of h_α. This amounts to $k = -1$, i.e. the space $L_{\beta-\alpha}$ has to be non-zero. □

5.6 Bases, Weyl chambers, and the Weyl group

■ We fix: L – a fssc Lie algebra, $H \subset L$ – a choice of CSA, and we denote: $\Phi = \Phi(L,H)$ – the root system of L, and $E = E(L,H)$ – the real span of the roots of L.

Definition 5.6.1. The *rank of L* is $\mathrm{rk}(L) := \dim_{\mathbb{C}} H$ $(= \dim_{\mathbb{R}} E(L,H))$.

■ For info: One can show that the rank is independent of the choice of CSA, that is, any two CSAs $H, H' \subset L$ have the same dimension. What is more, there is a Lie algebra automorphism of L which maps H to H'. In this sense, all choices of CSA are equivalent. (Humphreys, Introduction to Lie algebras and representation theory, Section 16.4.)

Definition 5.6.2. 1. A subset $\Delta \subset \Phi$ is a *base of* Φ if it satisfies

B1) Δ is a basis of E (over $\mathbb{R}$), and

B2) each $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$, where $k_\alpha \in \mathbb{Z}$, and either all $k_\alpha \geq 0$, or all $k_\alpha \leq 0$.

2. The elements in Δ are called *simple roots*.

3. An element $\beta \in \Phi$ is called a *positive root* if all k_α in B2 are positive, and a *negative root* if all k_α are negative. We write

$$\Phi^+ = \{\beta \in \Phi \,|\, \beta \text{ is positive}\} \quad , \quad \Phi^- = \{\beta \in \Phi \,|\, \beta \text{ is negative}\} \ .$$

(By construction, $\Phi = \Phi^+ \dot\cup \Phi^-$, where $\dot\cup$ indicates that this is a union of disjoint sets.)

■ Note that by Theorem 5.5.6 (2), Δ is also a $\mathbb{C}$-basis of H^*.

Example 5.6.3. *A base of the root system of $sl(n,\mathbb{C})$:*
Recall from Example 5.5.7: the CSA we chose consists of trace-less diagonal matrices, so that $\mathrm{rk}(sl(n,\mathbb{C})) = n-1$, and the root system can be written as

$$\Phi = \{\alpha_{ij} := \varepsilon_i - \varepsilon_j \,|\, i,j = 1,\dots,n \ , \ i \neq j\} \ .$$

Claim: The set $\Delta = \{\alpha_{i,i+1} \,|\, i = 1,\dots,n-1\}$ is a base of Φ.

⌈ B1 is clear. For B2 note that for all $i \neq j$,

$$\alpha_{ij} = \begin{cases} \sum_{k=i}^{j-1} 1 \cdot \alpha_{k,k+1} & ; i < j \\ \sum_{k=i}^{j-1} (-1) \cdot \alpha_{k,k+1} & ; i > j \end{cases} \ .$$

Thus the k_α from B2 are 0 or 1 in the first case, and 0 or -1 in the second case. ⌋

Lemma 5.6.4. Let Δ be a base and $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$. Then $(\alpha,\beta) \leq 0$.

Proof. Suppose that $(\alpha, \beta) > 0$. By Lemma 5.5.8 it follows that $\alpha - \beta \in \Phi$. But this has coefficients $+1$ and -1, in contradiction to B2. $\square$

- For $\alpha \in \Phi$ denote the hyperplane orthogonal to α by

$$P_\alpha := \{x \in E \,|\, (x, \alpha) = 0\} \ .$$

The elements of $E \setminus \bigcup_{\alpha \in \Phi} P_\alpha$ are called *regular elements of E*. We have

$$\gamma \text{ regular} \quad \Leftrightarrow \quad \forall \alpha \in \Phi : (\gamma, \alpha) \neq 0 \ .$$

The connected components of $E \setminus \bigcup_{\alpha \in \Phi} P_\alpha$ are called *(open) Weyl chambers*.

- For a regular element γ we set

$$\begin{aligned} C(\gamma) &:= \{x \in E \,|\, \forall \alpha \in \Phi : \text{if } (\gamma, \alpha) > 0 \text{ (resp. } < 0) \\ &\qquad\qquad\qquad \text{then } (x, \alpha) > 0 \text{ (resp. } < 0)\} \ , \\ \Phi^\pm(\gamma) &:= \{\alpha \in \Phi \,|\, \pm(\gamma, \alpha) > 0\} \ , \\ \Delta(\gamma) &:= \{\alpha \in \Phi^+(\gamma) \,|\, \alpha \neq \beta_1 + \beta_2 \text{ for all } \beta_1, \beta_2 \in \Phi^+(\gamma)\} \ . \end{aligned}$$

Note that $C(\gamma)$ is precisely the Weyl chamber containing γ. Furthermore, $\Phi^+(\gamma)$ and $\Phi^-(\gamma)$ are disjoint sets, and each $\alpha \in \Phi$ is contained either in $\Phi^+(\gamma)$ or in $\Phi^-(\gamma)$ (as $(\gamma, \alpha) = 0$ is excluded by regularity of γ),

$$\Phi = \Phi^+(\gamma) \,\dot\cup\, \Phi^-(\gamma) \ .$$

Theorem 5.6.5.
1. For γ regular, $\Delta(\gamma)$ is a base of Φ, and $\Phi^\pm(\gamma)$ are the corresponding sets of positive and negative roots.
2. For each base Δ of Φ, there is a regular γ such that $\Delta = \Delta(\gamma)$.

Proof. Part 1:

Claim 1: Each $\alpha \in \Phi^+(\gamma)$ is a $\mathbb{Z}_{\geq 0}$-linear combination of elements in $\Delta(\gamma)$.

⌈ Suppose there is an $\alpha \in \Phi^+(\gamma)$ that is not a $\mathbb{Z}_{\geq 0}$-linear combination. Then in particular $\alpha \notin \Delta(\gamma)$ (or $\alpha = 1 \cdot \alpha$). Amongst all such α, choose one for which (γ, α) is minimal. By assumption, $\alpha = \beta_1 + \beta_2$ with $\beta_1, \beta_2 \in \Phi^+(\gamma)$.

Since $(\gamma, \alpha) = (\gamma, \beta_1) + (\gamma, \beta_2)$ with both summands on the right hand side positive, by minimality of (γ, α) it follows that β_1, β_2 are $\mathbb{Z}_{\geq 0}$-linear combinations of $\Phi^+(\gamma)$. Then also α is a $\mathbb{Z}_{\geq 0}$-linear combination of $\Phi^+(\gamma)$, contradiction. ⌋

Claim 2: Any two distinct elements $\alpha, \beta \in \Delta(\gamma)$ satisfy $(\alpha, \beta) \leq 0$.

⌈ Suppose $(\alpha, \beta) > 0$. By Lemma 5.5.8, also $\alpha - \beta \in \Phi$. But $\alpha - \beta \notin \Phi^+(\gamma)$ (as then $\alpha = \beta + (\alpha - \beta)$, in contradiction to $\alpha \in \Delta(\gamma)$) and $\alpha - \beta \notin \Phi^-(\gamma)$ (as then $\beta - \alpha \in \Phi^+(\gamma)$ and $\beta = \alpha + (\beta - \alpha)$, in contradiction to $\beta \in \Delta(\gamma)$). This is a contradiction to $\Phi = \Phi^+(\gamma) \cup \Phi^-(\gamma)$. ⌋

For the next claim we need the following fact from linear algebra (Exercise): "Let F be a euclidean space, and let $v_1, \dots, v_m$ be vectors in F such that $(v_i, v_j) \leq 0$ for all $i \neq j$. Suppose that all v_i lie on the same side of some hyperplane (i.e. there is an $x \in F$ such that $(v_i, x) > 0$ for all i). Then $v_1, \dots, v_m$ are linearly independent." E (Ex. 36)

Claim 3: $\Delta(\gamma)$ is a base.

⌈ By definition, $\Delta(\gamma) \subset \Phi^+(\gamma)$. Then the above fact (with $x = \gamma$) together with claim 2 shows that $\Delta(\gamma)$ is linearly independent. Furthermore

$$E = \mathrm{span}_{\mathbb{R}}\Phi = \mathrm{span}_{\mathbb{R}}\Phi^+(\gamma) \overset{\text{Claim 1}}{=} \mathrm{span}_{\mathbb{R}}\Delta(\gamma) \ ,$$

so that $\Delta(\gamma)$ is in fact a basis of E, establishing property B1 of a base. Property B2 follows as $\Phi = \Phi^+(\gamma) \,\dot{\cup}\, \Phi^-(\gamma)$, and elements of $\Phi^+(\gamma)$ are $\mathbb{Z}_{\geq 0}$-linear combinations of $\Delta(\gamma)$ by claim 1, and $\Phi^-(\gamma) = -\Phi^+(\gamma)$. ⌋

Claim 1 shows that all elements in $\Phi^+(\gamma)$ are positive roots for $\Delta(\gamma)$. As by construction $\Phi^-(\gamma) = -\Phi^+(\gamma)$, it follows that all elements in $\Phi^-(\gamma)$ are the negative roots. But $\Phi = \Phi^+(\gamma) \,\dot{\cup}\, \Phi^-(\gamma)$, which shows that $\Phi^\pm(\gamma)$ is the complete set of positive/negative roots.

Part 2:

Claim: The set $D := \{x \in E \,|\, (x, \alpha) > 0 \text{ for all } \alpha \in \Delta\}$ is not empty.

⌈ Let $d_\alpha \in E$ ($\alpha \in \Delta$) be the basis dual to Δ in the sense that for all $\alpha, \beta \in \Delta(\gamma)$ we have $(\alpha, d_\beta) = \delta_{\alpha,\beta}$. Define

$$S := \left\{ \textstyle\sum_{\alpha \in \Delta} s_\alpha d_\alpha \,\middle|\, r_\alpha > 0 \right\} .$$

Clearly, $S \neq \emptyset$. Furthermore, $S \subset D$ since for any $x \in S$ and $\alpha \in \Delta$, we have

$$(x, \alpha) = \sum_{\beta \in \Delta} s_\beta (d_\beta, \alpha) = s_\alpha > 0 \ .$$

⌋

By the above claim, we can choose $\gamma \in D$. Then:

- γ is regular: We need to check that $(\gamma, \beta) \neq 0$ for all $\beta \in \Phi$, not just for $\beta \in \Delta$ (which holds by construction). Suppose $\beta \in \Phi^+$. Then $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ for some $k_\alpha \in \mathbb{Z}_{\geq 0}$. We get $(\gamma, \beta) = \sum_{\alpha \in \Delta} k_\alpha (\gamma, \alpha) > 0$, as $(\gamma, \alpha) > 0$ by construction and not all k_α can be zero.

  For $\beta \in \Phi^-$ we analogously find $(\gamma, \beta) < 0$.

- $\Phi^\pm \subset \Phi^\pm(\gamma)$: Clear, as in the previous point we checked that $\pm(\gamma, \alpha) > 0$ for $\alpha \in \Phi^\pm$.

- $\Phi^\pm = \Phi^\pm(\gamma)$: Follows from the previous point as $\Phi = \Phi^+ \dot\cup \Phi^-$ and $\Phi = \Phi^+(\gamma) \dot\cup \Phi^-(\gamma)$.

- $\Delta \subset \Delta(\gamma)$: Suppose $\alpha \in \Delta$ but $\alpha \notin \Delta(\gamma)$. Then there exist $\beta_1, \beta_2 \in \Phi^+(\gamma) = \Phi^+$ such that $\alpha = \beta_1 + \beta_2$. Each of β_1, β_2 are $\mathbb{Z}_{\geq 0}$-linear combinations of elements in Δ where not all coefficients are zero. This is a contradiction to linear independence of Δ.

- $\Delta = \Delta(\gamma)$: Follows as both Δ and $\Delta(\gamma)$ form a basis of E and hence both have $\mathrm{rk}(L)$ many elements.

$\square$

Definition 5.6.6. Given a base Δ of Φ, the *fundamental Weyl chamber* is

$$C(\Delta) := C(\gamma) \quad \text{where } \gamma \text{ is regular s.t. } \Delta = \Delta(\gamma) \ .$$

Remark 5.6.7.

1. We have $C(\Delta) = \{x \in E \,|\, (x, \alpha) > 0 \text{ for all } \alpha \in \Delta\}$. (Why? *Hint:* Check that $C(\gamma) = \{x \in E \,|\, \forall \alpha \in \Phi^+(\gamma) : (x, \alpha) > 0\}$.) ?

2. There is a one-to-one correspondence

$$\begin{tikzcd} \{\text{Weyl chambers } C\} \arrow[r, bend left, "(1)"] & \{\text{bases } \Delta \text{ of } \Phi\} \arrow[l, bend left, "(2)"] \end{tikzcd} \quad ,$$

   where for the map (1) one picks any $\gamma \in C$ and takes the base $\Delta(\gamma)$, and for the map (2) one sends Δ to $C(\Delta)$.

(22.12.)

■ For each $\alpha \in \Phi$ define the *reflection along the root* α to be the linear map

$$s_\alpha : E \longrightarrow E \quad , \quad \lambda \longmapsto \lambda - 2 \, \tfrac{(\lambda, \alpha)}{(\alpha, \alpha)} \, \alpha \ .$$

We use the same notation and defining formula for $s_\alpha : H^* \to H^*$. Some properties of s_α are:

- $s_\alpha(\alpha) = -\alpha$,
- if $(\lambda, \alpha) = 0$, then $s_\alpha(\lambda) = \lambda$,
- $s_\alpha \circ s_\alpha = \mathrm{id}$,
- $s_\alpha = s_{-\alpha}$.

Geometrically, s_α is a reflection at the hyperplane P_α orthogonal to α (Why?). ?

Definition 5.6.8. The *Weyl group W of L* is the subgroup of $GL(E)$ generated by the s_α, $\alpha \in \Phi$. Explicitly,

$$W = \{ s_{\alpha_1} \circ \cdots \circ s_{\alpha_m} \,|\, m \geq 0\,,\ \alpha_i \in \Phi \} \ .$$

(There is no need to explicitly include inverses of the generators as $s_\alpha^{-1} = s_\alpha$.)

Remark 5.6.9. The Weyl group acts on E by isometries: For all $\lambda, \mu \in E$ and $w \in W$ we have $(w(\lambda), w(\mu)) = (\lambda, \mu)$ (Why?). In other words, ?

$$W \subset O(E) \ .$$

Theorem 5.6.10. For all $w \in W$ and $\beta \in \Phi$ also $w(\beta) \in \Phi$.

Proof. It is enough to show that for all $\alpha, \beta \in \Phi$ we have $s_\alpha(\beta) \in \Phi$.

The direct sum $M = \bigoplus_{k \in \mathbb{Z}} L_{\beta + k\alpha}$ is a (not necessarily irreducible) sl_α-module (because of Lemma 5.4.8 (1)). Recall from the proof of Theorem 5.5.5 that h_α acts on $L_{\beta+k\alpha}$ by multiplication with $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2k$. Since $\beta \in \Phi$ by assumption, we have $L_\beta \neq \{0\}$, and so $\lambda := \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an eigenvalue of h_α.

In Remark 5.5.3 (3) we saw that if λ is an eigenvalue of h in a finite-dimensional $sl(2,\mathbb{C})$-representation, then so is $-\lambda$. Thus also

$$-\lambda = -\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2k \quad \text{with} \quad k = -\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$$

is an h_α-eigenvalue, and so for this value of k we have $L_{\beta+k\alpha} \neq \{0\}$. Now

$$\beta + k\alpha = \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha = s_\alpha(\beta) \ .$$

Altogether we see that $L_{s_\alpha(\beta)} \neq \{0\}$ and so $s_\alpha(\beta) \in \Phi$. $\square$

Corollary 5.6.11. The Weyl group of L is finite.

Proof. An element $w \in W \subset O(E)$ is completely determined by its values on a spanning set of E. In particular, we w is determined by the values it takes on Φ. By Theorem 5.6.10, w restricts to a bijection $\Phi \to \Phi$ (both w and w^{-1} restrict to maps $\Phi \to \Phi$). But Φ is a finite set, and the number of bijections $\Phi \to \Phi$ is given by $|\Phi|!$ (the number of elements in the corresponding permutation group). Thus also $|W| \leq |\Phi|!$. □

Example 5.6.12. *The Weyl group of $sl(n, \mathbb{C})$:*
We start by recalling some notation from Example 5.5.7. The root space is

$$\Phi = \{\alpha_{ij} := \varepsilon_i - \varepsilon_j \,|\, i, j = 1, \dots, n \ , \ i \neq j\} \ ,$$

where the ε_i satisfy $\sum_i \varepsilon_i = 0$. We can still write $\lambda \in H^*$ uniquely in terms of ε_i as $\lambda = \sum_{j=1}^n \lambda_j \varepsilon_j$ provided we demand $\sum_{j=1}^n \lambda_j = 0$. In this convention, the bilinear form $(-,-)$ on H^* is given by

$$(\lambda, \mu) = \frac{1}{2n} \sum_{j=1}^n \lambda_j \mu_j \ .$$

We can now compute the reflection along the root α_{ij}:

$$\begin{aligned} s_{\alpha_{ij}}(\lambda) &= \lambda - 2\frac{(\lambda, \alpha_{ij})}{(\alpha_{ij}, \alpha_{ij})}\alpha_{ij} = \lambda - 2\frac{\frac{1}{2n}(\lambda_i - \lambda_j)}{\frac{1}{2n}\,2}(\varepsilon_i - \varepsilon_j) \\ &= \sum_{k \neq i,j} \lambda_k \varepsilon_k \ + \ \lambda_j \varepsilon_i + \lambda_i \varepsilon_j \ . \end{aligned}$$

Thus, $s_{\alpha_{ij}}$ acts on the coefficients of λ by a transposition which exchanges the coefficients of ε_i and ε_j (note that this does not affect the condition $\sum_k \lambda_k = 0$). Since all values $i \neq j$ are allowed, these transpositions generate the permutation group S_n of n elements, and we have a group isomorphism $S_n \to W$, $\sigma \mapsto w_\sigma$, where

$$w_\sigma(\lambda) = \sum_{k=1}^n \lambda_{\sigma^{-1}(k)} \, \varepsilon_k \ .$$

(Why does one need σ^{-1} for this to be a group homomorphism?) ?

■ For the rest of this subsection we fix a base Δ of Φ. For a simple root α (i.e. for $\alpha \in \Delta$), the corresponding reflection s_α is also called a *simple reflection*.

Lemma 5.6.13. Let $\alpha \in \Delta$. Then s_α permutes the set $\Phi^+ \setminus \{\alpha\}$.

Proof. Let $\beta \in \Phi^+$ such that $\beta \neq \alpha$. Note that then also $\beta \notin \mathbb{C}\alpha$ (the only roots proportional to α are $\pm\alpha$, and as $\alpha \in \Delta \subset \Phi^+$ we have $-\alpha \in \Phi^-$).

As $\beta \in \Phi^+$, we have

$$\beta = \sum_{\gamma \in \Delta} k_\gamma \, \gamma = \sum_{\gamma \in \Delta \setminus \{\alpha\}} k_\gamma \, \gamma \, + \, k_\alpha \, \alpha \quad \text{with} \quad k_\gamma \in \mathbb{Z}_{\geq 0} \ .$$

As $\beta \notin \mathbb{C}\alpha$, there must be at least one $\gamma \in \Delta \setminus \{\alpha\}$ such that $k_\gamma > 0$. Acting with s_α on β changes β by a multiple of α:

$$s_\alpha(\beta) = \sum_{\gamma \in \Delta \setminus \{\alpha\}} k_\gamma \, \gamma \, + \, (\text{const}) \, \alpha \ .$$

But since at least one of the k_γ in the first sum is strictly positive, by property (B2) of a base (Defintion 5.6.2) we have $s_\alpha(\beta) \in \Phi^+$. Furthermore, for the same reason, $s_\alpha(\beta) \notin \mathbb{C}\alpha$, so that in fact $s_\alpha(\beta) \in \Phi^+ \setminus \{\alpha\}$.

Altogether, s_α restricts to a map $\Phi^+ \setminus \{\alpha\} \to \Phi^+ \setminus \{\alpha\}$, and since it squares to the identity, it is a bijection. □

Corollary 5.6.14. Set $\delta = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$. Then for a simple root $\alpha \in \Delta$ we have $s_\alpha(\delta) = \delta - \alpha$.

Proof. We compute

$$s_\alpha(\delta) \ = \ \tfrac{1}{2} s_\alpha(\alpha) + \tfrac{1}{2} \sum_{\beta \in \Phi^+ \setminus \{\alpha\}} s_\alpha(\beta) \ \overset{(*)}{=} \ -\tfrac{1}{2}\alpha + \tfrac{1}{2} \sum_{\beta' \in \Phi^+ \setminus \{\alpha\}} \beta' \ = \ \delta - \alpha \ ,$$

where in step $(*)$ we used $s_\alpha(\alpha) = -\alpha$ and that by Lemma 5.6.13, s_α merely permutes the summands of $\sum_{\beta \in \Phi^+ \setminus \{\alpha\}}$ and can hence be absorbed into a redefinition of the summation variable. □

Theorem 5.6.15.

1. For all $\gamma \in E$ there is a $w \in W$ such that $(w(\gamma), \alpha) \geq 0$ for all $\alpha \in \Delta$. If γ is in addition regular, then $w(\gamma) \in C(\Delta)$.
2. If Δ' is another base for Φ, then there is $w \in W$ such that $w(\Delta') = \Delta$.
3. For each $\alpha \in \Phi$ there is $w \in W$ such that $w(\alpha) \in \Delta$.
4. W is generated by simple reflections, i.e. by $\{s_\alpha \,|\, \alpha \in \Delta\}$.

Proof. Let W' be the subgroup of W generated by simple reflections,

$$W' := \{ s_{\alpha_1} \circ \cdots \circ s_{\alpha_m} \,|\, m \geq 0 \, , \ \alpha_i \in \Delta \} \ \subset W \subset O(E) \ .$$

We will first show parts 1–3 of the theorem for the subgroup W' and then in part 4 we will show that actually $W' = W$.

Part 1 for W': Recall the definition of δ from Corollary 5.6.14 and note that $\{(w(\gamma),\delta)\,|\,w \in W'\}$ is a finite subset of $\mathbb{R}$. We can hence pick a $w \in W'$ such that $(w(\gamma),\delta)$ is maximal.

For all $\alpha \in \Delta$ also $s_\alpha \circ w \in W'$. By definition of w we get the inequality

$$(w(\gamma),\delta) \;\geq\; (s_\alpha \circ w(\gamma),\delta) \;\overset{(1)}{=}\; (s_\alpha \circ s_\alpha \circ w(\gamma), s_\alpha(\delta)) \;\overset{(2)}{=}\; (w(\gamma),\delta-\alpha)\ ,$$

where step 1 follows as s_α is an isometry, and step 2 uses $s_\alpha^2 = \mathrm{id}$ and Corollary 5.6.14. It follows that

$$(w(\gamma),\alpha) \geq 0 \quad \text{for all} \quad \alpha \in \Delta\ .$$

Furthermore, for regular γ we have $(w(\gamma),\alpha) > 0$ for all $\alpha \in \Delta$ (Why? *Hint:* Theorem 5.6.10), i.e. $w(\gamma) \in C(\Delta)$.

Part 2 for W': By Theorem 5.6.5 (2) there exists a regular $\gamma' \in E$ such that $\Delta' = \Delta(\gamma')$. By part 1 we can choose $w \in W'$ such that $w(\gamma') \in C(\Delta)$. But for any $\gamma \in C(\Delta)$ we have $\Delta = \Delta(\gamma)$, and so $\Delta = \Delta(w(\gamma'))$. Using this, we compute

$$w(\Delta') \;=\; w(\Delta(\gamma')) \;\overset{(*)}{=}\; \Delta(w(\gamma')) \;=\; \Delta\ .$$

(Why does $(*)$ hold? *Hint:* First show that for all regular $\psi \in E$ we have $w(\Phi^+(\psi)) = \Phi^+(w(\psi))$.)

(9.1.)

Part 3 for W': By part 2 it is enough to show that $\alpha \in \Delta(\gamma)$ for some regular γ. To construct such a γ, we start by choosing $\tilde\gamma \in P_\alpha$ such that $\gamma \notin P_\beta$ for all $\beta \in \Phi \setminus \{\pm\alpha\}$. This is possible as each P_β for $\beta \notin \mathbb{R}\alpha$ intersects P_α in a proper subspace and the union of a finite number of proper subspaces cannot give all of P_α. Thus $\tilde\gamma$ satisfies

$$(\tilde\gamma,\alpha) = 0 \quad \text{and} \quad (\tilde\gamma,\beta) \neq 0 \text{ for all } \beta \neq \pm\alpha\ .$$

By continuity, there exists $t,\varepsilon > 0$ such that for $\gamma := \tilde\gamma + t\alpha$ we have

$$(\gamma,\alpha) < \varepsilon \quad \text{and} \quad |(\gamma,\beta)| > \varepsilon \text{ for all } \beta \neq \pm\alpha\ . \tag{$*$}$$

(Details?) By construction, γ is regular, and as $(\gamma,\alpha) > 0$ we have $\alpha \in \Phi^+(\gamma)$. It remains to show that also $\alpha \in \Delta(\gamma)$, i.e. that there are no $\beta_1,\beta_2 \in \Phi^+(\gamma)$ with $\alpha = \beta_1 + \beta_2$. Suppose there were such β_1,β_2. Then, firstly, $\beta_1,\beta_2 \neq \pm\alpha$ (Why?), and, secondly,

$$\underbrace{(\gamma,\alpha)}_{<\varepsilon} \;=\; \underbrace{(\gamma,\beta_1)}_{>\varepsilon} + \underbrace{(\gamma,\beta_2)}_{>\varepsilon}\ .$$

Here we used $(*)$ and the fact that $\beta_1, \beta_2 \in \Phi^+(\gamma)$ so that $(\gamma, \beta_j) > 0$. Altogether we obtain a contradiction, so that indeed $\alpha \in \Delta(\gamma)$.

Part 4: We will show that for all $\beta \in \Phi$ we have $s_\beta \in W'$ (and not just for $\beta \in \Delta$). This then implies $W' = W$.

Let $\beta \in \Phi$ be given. By part 3, there $w \in W'$ such that $w(\beta) \in \Delta$. Thus $s_{w(\beta)} \in W'$. But

$$s_{w(\beta)} \;=\; w \circ s_\beta \circ w^{-1} \; .$$

(Why? *Hint:* Just write out both sides and use that w is an isometry.) ? Rewriting this as $s_\beta = w^{-1} \circ s_{w(\beta)} \circ w$, we see that all maps on the right hand side are in W', and so also $s_\beta \in W'$. □

Remark 5.6.16.

1. Parts 1 and 2 of the above theorem say that the Weyl group W acts transitively on the set of Weyl chambers, and on the set $\{\Delta | \Delta \text{ is a base for } \Phi\}$.

   For Info: In fact, W acts simply transitively on the set of Weyl chambers and the set of bases. (Humphreys, Introduction to Lie algebras and representation theory, Section 10.3 Theorem e.)

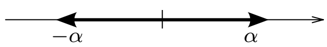
2. By part 3 of the theorem, the W-orbit of the elements in any given base Δ is all of Φ, i.e. $\{w(\alpha) \,|\, \alpha \in \Delta, w \in W\} = \Phi$.

5.7 Classification via Dynkin diagrams

■ In this subsection we fix a fssc Lie algebra L and a CSA $H \subset L$. Note that the dimension of L is given by (Why?) ?

$$\dim_{\mathbb{C}} L \;=\; \mathrm{rk}(L) + |\Phi| \; .$$

Example 5.7.1. *Classification of root systems of rank 1 Lie algebras:*
In this case, there has to be at least one root $\alpha \in \Phi$ (to get a non-zero span E), and as E is 1-dimensional, any two roots are proportional. As with α also $-\alpha$ is a root (Lemma 5.5.1 (1)), and by Theorem 5.5.5 (1) only the roots $\pm\alpha$ are allowed, $\Phi = \{\alpha, -\alpha\}$:



By the above formula, $\dim_{\mathbb{C}} L = 1 + 2 = 3$ On the other hand, by Proposition 5.5.2, for each root α we get an injective Lie algebra homomorphism $sl(2,\mathbb{C}) \to L$, and by dimensional reasons, this has to be an isomorphism.

We conclude that up to isomorphism, there is precisely one fssc Lie algebra of rank 1, namely $sl(2,\mathbb{C})$. This Lie algebra is also called A_1.

Lemma 5.7.2. Let $\alpha, \beta \in \Phi$ such that $\beta \neq \pm\alpha$ and $(\alpha, \beta) > 0$. Suppose $(\beta, \beta) \geq (\alpha, \alpha)$. Set

$$p = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}_{>0} \quad , \quad q = \frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}_{>0} \ .$$

Then $p \in \{1, 2, 3\}$ and $q = 1$.

Proof. By the Cauchy-Schwartz inequality, for all $u, v \in E$ we have

$$(u, v)^2 \leq (u, u)(v, v) \ ,$$

and $(u, v)^2 = (u, u)(v, v)$ iff u and v are colinear. By assumption, $\beta \neq \pm\alpha$, and hence α, β are not colinear (Theorem 5.5.5 (1)). Thus

$$(\alpha, \beta)^2 < (\alpha, \alpha)\,(\beta, \beta) \ ,$$

and hence $pq < 4$. Furthermore, we have $p, q \in \mathbb{Z}_{>0}$, and by assumption $p > q$. The only possible values for the pair (p, q) are therefore $(1, 1)$, $(2, 1)$, $(3, 1)$. □

Remark 5.7.3. The angle $\psi \in [0, \frac{\pi}{2}]$ between the lines $\mathbb{R}\alpha$ and $\mathbb{R}\beta$ is

$$\big(\cos\psi\big)^2 \ = \ \frac{(\alpha, \beta)^2}{(\alpha, \alpha)\,(\beta, \beta)} \ = \ \frac{pq}{2} \ .$$

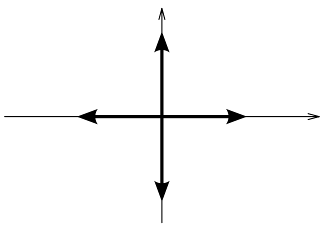
If $(\alpha, \beta) \neq 0$, the ratio of the squared length of the roots is $(\beta, \beta)/(\alpha, \alpha) = p/q$. From Lemma 5.7.2, we obtain the following table of possible angles and length ratios between roots $\beta \neq \pm\alpha$:

p	q	ψ	$\frac{(\beta,\beta)}{(\alpha,\alpha)}$
1	1	60°	1
2	1	45°	2
3	1	30°	3
0	0	90°	no cond.

The last line is the case $(\alpha, \beta) = 0$.

Example 5.7.4. *Classification of root systems of rank 2 Lie algebras:*
Since E is 2-dimensional, Φ contains two roots which are non-colinear. non-colinear roots. Let θ_m be the smallest angle between any pair of non-colinear roots in Φ. Since with α also $-\alpha$ is a root, we have $\theta_m \in (0, \frac{\pi}{2}]$. According to Remark 5.7.3, there are only four possible values for θ_m:

- $\theta_m = 90°$: The root system is

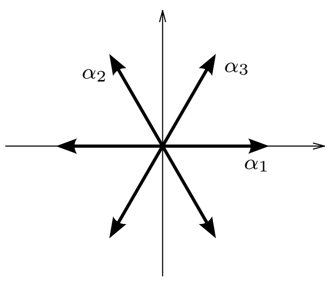


These are two orthogonal copies of the root system for $sl(2, \mathbb{C})$, and in this case it follows together with Example 5.7.1 that $L \cong sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C})$ (Exercise). E (Ex. 38)

- $\theta_m = 60°$: Let α_1, α_3 be two roots with this angle. By the table in Remark 5.7.3, α_1 and α_3 have the same length. By Theorem 5.6.10, also

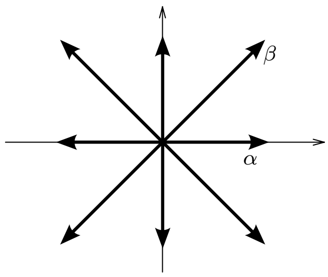
$$s_{\alpha_3}(\alpha_1) = \alpha_1 - \underbrace{2\frac{(\alpha_1, \alpha_3)}{(\alpha_3, \alpha_3)}}_{=p=1} \alpha_3 = \alpha_1 - \alpha_3 =: \alpha_2$$

is a root. The root system thus contains at least six roots, $\Phi \supset \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3\}$. If there were an additional root β, it cannot be proportional to α_i, $i = 1, 2, 3$. But then it would form an angle of less than $60°$ with one of the α_i, contradicting the assumption on θ_m. We arrive at the following root system:



This is the root system of $L = sl(3, \mathbb{C})$ (Example 5.5.7), which is also called A_2. We will later state (but not prove) that the root system determines a fssc Lie algebra up to isomorphism.

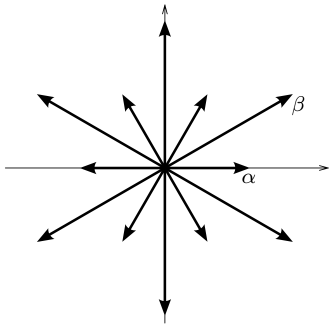
- $\theta_m = 45°$: Let α, β be two roots with this angle. Then by the above table, $(\beta, \beta) = 2(\alpha, \alpha)$. Completing with respect to Weyl reflections gives: (Details?) ?



This is the root system of $L = sp(4, \mathbb{C})$ (Exercise), also called C_2. E

(Ex. 39)

- $\theta_m = 30°$: If α, β are two roots with this angle, the above table gives $(\beta, \beta) = 3(\alpha, \alpha)$. Completing with respect to Weyl reflections gives



(Details?) This Lie algebra is called G_2. (See Fulton, Harris, Representation theory – A first course, Chapter 22 for more on G_2.) ?

Definition 5.7.5. Given a base Δ of Φ and an ordering $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ of the simple roots, the *Cartan matrix* is the $r \times r$ matrix A with entries

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \ .$$

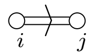
Remark 5.7.6. The Cartan matrix satisfies

- $A_{ij} \in \mathbb{Z}$ (Theorem 5.5.5 (3)),
- $A_{ii} = 2$ and $A_{ij} \leq 0$ if $i \neq j$ (Lemma 5.6.4),
- $A_{ij}A_{ji} \in \{0, 1, 2, 3\}$ if $i \neq j$ (Lemma 5.7.2).

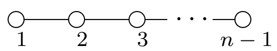
It is also clear from the definition that $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$. Using the properties above, we also see that, if $A_{ij} \neq 0$ for $i \neq j$, then $A_{ij} \in \{-1, -2, -3\}$, and if $A_{ij} \in \{-2, -3\}$ then $A_{ji} = -1$.

■ A *Dynkin diagram* is a pictorial representation of a Cartan matrix obtained as follows:

1. Draw dots (called vertices) labelled $1, \dots, r$.
2. For $i \neq j$ draw $A_{ij}A_{ji}$ lines between the vertices i and j.
3. If $|A_{ij}| > |A_{ji}|$ draw an arrowhead '>' on the lines between i and j pointing from i to j.

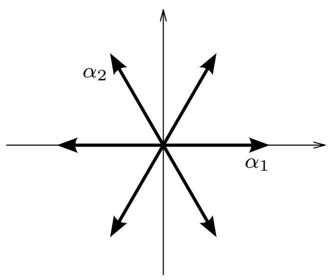
Remark 5.7.7.
1. If there is an arrow from node i to node j  then
$$\frac{(\alpha_i, \alpha_i)}{(\alpha_j, \alpha_j)} = \frac{|A_{ij}|}{|A_{ji}|} > 1 \quad ,$$
i.e. the root α_i is longer than the root α_j.
2. The labeling of the vertices by $1, \dots, r$ is not part of the Dynkin diagram. Instead it encodes the ordering of the simple roots, which is an additional choice.

Example 5.7.8. *Cartan matrix and Dynkin diagram for $sl(n, \mathbb{C})$:*
Recall the root system $\Phi = \{\alpha_{ij} := \varepsilon_i - \varepsilon_j \,|\, i, j = 1, \dots, n \;,\; i \neq j\}$, and from Example 5.6.3 that a base of Φ is given by
$$\Delta = \{\alpha_i := \alpha_{i,i+1} \,|\, i = 1, \dots, n-1\} \; .$$
Using the explicit inner product from Example 5.5.7, we find
$$(\alpha_i, \alpha_i) = \tfrac{1}{n} \quad , \quad (\alpha_i, \alpha_{i+1}) = -\tfrac{1}{2n} \quad , \quad (\alpha_i, \alpha_{i+j}) = 0 \text{ for } j \geq 2 \; .$$
This gives
$$A_{ij} \;=\; \begin{cases} 2 & ; i = j \\ -1 & ; |i-j| = 1 \\ 0 & ; \text{else} \end{cases} \quad .$$
The corresponding Dynkin diagram is (12.1.)

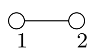


Example 5.7.9. *Cartan matrix and Dynkin diagram for rank 2:*
Below we list the choice of base $\Delta = \{\alpha_1, \alpha_2\}$ together with the corresponding Cartan matrix and Dynkin diagram.

1. $A_2 = sl(3, \mathbb{C})$: Already treated in the previous example,



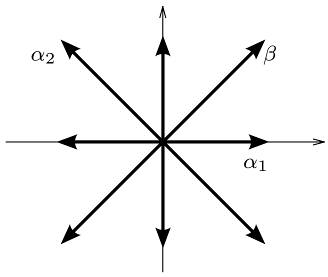
$$, \quad A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad ,$$



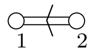
2. $C_2 = sp(4,\mathbb{C})$: In Example 5.7.4 we saw that $p = \frac{2(\alpha_1,\beta)}{(\alpha_1,\alpha_1)} = 2$ and $q = \frac{2(\alpha_1,\beta)}{(\beta,\beta)} = 1$. Using the symmetries in the root system gives

$$A_{12} = \frac{2(\alpha_1,\alpha_2)}{(\alpha_2,\alpha_2)} = -1 \quad , \quad A_{21} = \frac{2(\alpha_2,\alpha_1)}{(\alpha_1,\alpha_1)} = -2 \ .$$

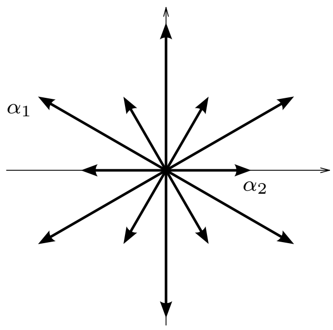
Thus:



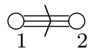
$$, \quad A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad ,$$



3. G_2: Analogously, we get (Details? Why do α_1, α_2 form a base?) ?



$$, \quad A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \quad ,$$



Remark 5.7.10. 1. A root system is called *irreducible*, if one cannot write $\Phi = \Phi_1 \dot\cup \Phi_2$ with non-empty Φ_1, Φ_2 such that $(\gamma_1, \gamma_2) = 0$ for all $\gamma_1 \in \Phi_1$, $\gamma_2 \in \Phi_2$.

In Example 5.7.4, the first root system is not irreducible, and the remaining three are irreducible.

2. The following statements on the fssc Lie algebra L are equivalent:

   (a) L is simple.

   (b) Φ is irreducible.

   (c) The Dynkin diagram is connected.

   (Exercise.) E

(Ex. 38, 40, 43)

■ The remaining theorems in this section we will just state but not prove.

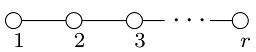
Theorem 5.7.11. Let L, L' be fssc Lie algebras. The following are equivalent:

1. $L \cong L'$ as Lie algebras.
2. For any choice of CSA $H \subset L$ and $H' \subset L'$, there exists an $s \in \mathbb{R}_{>0}$ and an isometry $\psi : E(L, H) \to E(L', H')$ such that
$$\psi(\Phi(L, H)) = s \cdot \psi(\Phi(L', H'))$$

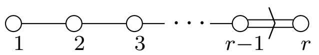
(For a proof see Humphreys, Section 14.2.)

Remark 5.7.12.
1. In words, the theorem says that two fssc Lie algebras are isomorphic if and only if their root systems are isometric, up to a possible rescaling. Of course, the scale factor then has to be $s = 1$ (as by part 1 $L \cong L'$), but the current formulation is useful because of the next point.
2. If L is in addition simple, so that the Dynkin diagram is connected, the corresponding Cartan matrix determines Φ up to an overall rescaling. Thus: Two finite-dimensional simple complex Lie algebras are isomorphic if and only if their (unlabelled) Dynkin diagrams agree.

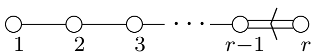
Theorem 5.7.13 (Killing-Cartan-Dynkin)**.** Let L be a simple finite-dimensional complex Lie algebra. Choose a CSA $H \subset L$ and a base Δ of $\Phi(L, H)$. The Dynkin diagram of L with respect to H, Δ is one of the following (the labelling of the vertices is not part of the diagram):

A_r = 

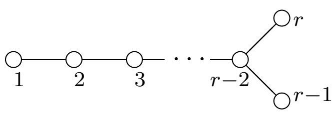
for $r \geq 1$

B_r = 

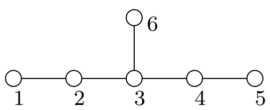
for $r \geq 2$

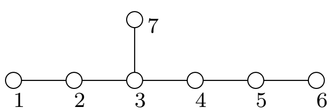
C_r = 

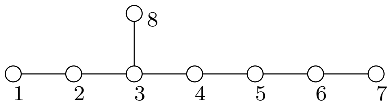
for $r \geq 3$

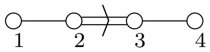
D_r = 

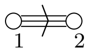
for $r \geq 4$

E_6 = 

E_7 = 

E_8 = 

F_4 = 

G_2 = 

Furthermore, each of the above diagrams occurs for some choice of L.

(For a proof see Humphreys, Sections 11.4 and 18.4.)

Remark 5.7.14. 1. The names of the Dynkin diagrams above are also used to denote the corresponding Lie algebras. The index gives the rank of that Lie algebra, which is the same as the number of vertices in the Dynkin diagram. E.g. A_r has rank r.

2. We have already met the four infinite series under different names (but we only proved that fact for A_r):

   - $A_r = sl(r+1, \mathbb{C})$
   - $B_r = so(2r+1, \mathbb{C})$
   - $C_r = sp(2r, \mathbb{C})$
   - $D_r = so(2r, \mathbb{C})$

   These are called *classical Lie algebras*. The remaining 5 cases E_6, E_7, E_8, F_4, G_2 are called *exceptional Lie algebras*.

The restrictions placed on r in the classification theorem ensure that each Dynkin diagram appears exactly once. For small values of r some Dynkin diagrams coincide and give isomorphisms of Lie algebras:

$$B_2 = so(5, \mathbb{C}) \cong sp(4, \mathbb{C}) = C_2 \quad , \quad D_3 = so(6, \mathbb{C}) \cong sl(4, \mathbb{C}) = A_3 \ .$$

6 Representations of semisimple Lie algebras

In this section we fix

- L: fssc Lie algebra,
- $H \subset L$: a choice of CSA,
- $\Delta \subset \Phi \equiv \Phi(L, H)$: a choice of base,

- $\Delta = \{\alpha_1, \dots, \alpha_r\}$: a choice of ordering of the simple roots.

This results into a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots, and defines the euclidean space $E = E(L, H) \subset H^*$.

6.1 The weight lattice

■ A subset $\Gamma \subset V$ of a finite-dimensional vector space V is called a *lattice* if there is a basis $\{v_1, \dots, v_n\}$ of V such that Γ is the $\mathbb{Z}$-linear span of the basis vectors,

$$\Gamma = \{m_1 v_1 + \cdots + m_n v_n \,|\, m_i \in \mathbb{Z}\} \ .$$

The following notions depend just on the choice of H, but not on the choice of base Δ:

- *Integral weight:* An element $\lambda \in H^*$ is called a *weight*. A weight λ is *integral* if $2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi$. The motivates the definition of the *coroot* $\alpha^\vee$:

  $$\text{for } \alpha \in \Phi : \quad \alpha^\vee := \frac{2}{(\alpha, \alpha)} \alpha \in H^* \ .$$

  Then $\lambda \in H^*$ is integral iff $(\lambda, \alpha^\vee) \in \mathbb{Z}$ for all $\alpha \in \Phi$.

- *Weight lattice:* The set of all integral weights is called the *weight lattice* Λ:

  $$\Lambda := \{\lambda \in H^* \,|\, \lambda \text{ is integral}\} \ .$$

  We will see in Remark 6.1.1 below that this is indeed a lattice. For now note that a $\mathbb{Z}$-linear combination of elements in Λ is again in Λ.

- *Root lattice:* The $\mathbb{Z}$-linear span of all roots is called the *root lattice* Λ_r:

  $$\Lambda_r := \{\textstyle\sum_{\alpha \in \Phi} m_\alpha \, \alpha \,|\, m_\alpha \in \mathbb{Z}\} \ .$$

  Again it is not clear that this is actually a lattice, but we will check this in Remark 6.1.1. Every root is an integral weight, $\Phi \subset \Lambda$ (Why?), ?
  and hence also

  $$\Lambda_r \subset \Lambda \ .$$

The next notions in addition depend on the choice of base $\Delta = \{\alpha_1, \dots, \alpha_r\}$:

- *Dominant weight:* An integral weight is called *dominant* if $(\lambda, \alpha_i^\vee) \in \mathbb{Z}_{\geq 0}$ for $i = 1, \dots, r$. We denote the set of dominant weights by

  $$\Lambda^+ := \{\lambda \in H^* \,|\, (\lambda, \alpha_i^\vee) \in \mathbb{Z}_{\geq 0} \text{ for } i = 1, \dots, r\} \ .$$

Clearly, $\Lambda^+ \subset \Lambda$, but Λ^+ is not a lattice (e.g. $x \in \Lambda^+$, $x \neq 0$ implies $-x \notin \Lambda^+$). Equivalently, λ is dominant iff it is integral and satisfies $(\lambda, \alpha_i) \geq 0$ for $i = 1, \dots, r$.

- *Fundamental dominant weight:* Let $\{\omega_1, \dots, \omega_r\}$ be the $\mathbb{R}$-basis of E dual to Δ in the sense that

$$(\omega_i, \alpha_j^\vee) = \delta_{i,j} \quad \text{for} \quad i, j \in \{1, \dots, r\} \ .$$

Note that the ω_i also form a $\mathbb{C}$-basis of H^*. By definition, $(\omega_i, \alpha_j^\vee) \in \mathbb{Z}_{\geq 0}$, so that $\omega_i \in \Lambda^+$. The ω_i are called *fundamental dominant weights*.

Remark 6.1.1. Parts 2 and 3 of this remark show that Λ and Λ_r are indeed lattices in E (and also in H^*).

1. $\lambda \in H^*$ is integral iff $(\lambda, \alpha_i^\vee) \in \mathbb{Z}$ for $i = 1, \dots, r$. (Why does $\Leftarrow$ hold? I.e. why is it enough to test integrality against simple roots only?) ?

2. *Claim:* $\Lambda = \{\sum_{i=1}^r m_i \omega_i \,|\, m_i \in \mathbb{Z}\}$.

   ⌈ Let $\lambda = \sum_{i=1}^r \lambda_i \omega_i$ with $\lambda_i \in \mathbb{C}$ be a general element of H^*. Then

$$\begin{aligned} \lambda \in \Lambda \ &\overset{1.}{\Leftrightarrow}\ (\lambda, \alpha_j^\vee) \in \mathbb{Z} \text{ for } j = 1, \dots, r \\ &\Leftrightarrow\ \sum_{i=1}^r \lambda_i \underbrace{(\omega_i, \alpha_j^\vee)}_{=\delta_{i,j}} \in \mathbb{Z} \text{ for } j = 1, \dots, r \\ &\Leftrightarrow\ \lambda_j \in \mathbb{Z} \text{ for } j = 1, \dots, r \end{aligned}$$

   ⌋

3. *Claim:* $\Lambda_r = \{\sum_{i=1}^r m_i \alpha_i \,|\, m_i \in \mathbb{Z}\}$.
   (Why does this hold? *Hint:* Recall property (B2) of a base.) ?

4. The dominant weights can be written in terms of the fundamental dominant weights as (Why?) ?

$$\Lambda^+ = \{\textstyle\sum_{i=1}^r m_i \omega_i \,|\, m_i \in \mathbb{Z}_{\geq 0}\} \ .$$

   Analogously, we define the subset $\Lambda_r^+ \subset \Lambda_r$ as

$$\Lambda_r^+ = \{\textstyle\sum_{i=1}^r m_i \alpha_i \,|\, m_i \in \mathbb{Z}_{\geq 0}\} \ .$$

   There is a notational pitfall here as in general

$$\Lambda_r^+ \neq \Lambda_r \cap \Lambda^+ \ .$$

   (Can you given an example?) ?

(16.1.)

5. The basis change between simple roots and fundamental dominant weights can be expressed via the Cartan matrix:

$$\alpha_i = \sum_{j=1}^{r} A_{ij}\,\omega_j \quad , \quad \omega_i = \sum_{j=1}^{r} (A^{-1})_{ij}\alpha_j \ .$$

(Why? *Hint:* To check the first equality it is enough to check the inner product with all simple coroots.) ?

Example 6.1.2. For $sl(n,\mathbb{C})$, the simple roots have inner products (Example 5.7.8)

$$(\alpha_i,\alpha_i) = \tfrac{1}{n} \quad , \quad (\alpha_i,\alpha_{i+1}) = -\tfrac{1}{2n} \quad , \quad (\alpha_i,\alpha_{i+j}) = 0 \text{ for } j \geq 2 \ .$$

Thus, the coroots are given by

$$\alpha_i^\vee = \frac{2}{(\alpha_i,\alpha_i)}\,\alpha_i = 2n\,\alpha_i \ .$$

Let us consider $n = 1$ and $n = 2$ in more detail.

1. $sl(2,\mathbb{C})$: The condition $(\omega_1,\alpha_1^\vee) = 1$ is solved by $\omega_1 = \frac{1}{2}\alpha_1$. The weight and root lattice are

$$\Lambda = \mathbb{Z}\omega_1 \quad , \quad \Lambda_r = 2\mathbb{Z}\omega_1 \ .$$

2. $sl(3,\mathbb{C})$: The condition $(\omega_i,\alpha_j^\vee) = \delta_{i,j}$ is solved by

$$\omega_1 = \tfrac{1}{3}\big(2\alpha_1 + \alpha_2\big) \quad , \quad \omega_2 = \tfrac{1}{3}\big(\alpha_1 + 2\alpha_2\big) \ .$$

As already seen in Remark 6.1.1 (4), conversely we have

$$\alpha_1 = A_{11}\omega_1 + A_{12}\omega_2 = 2\omega_1 - \omega_2 \ , \quad \alpha_2 = A_{21}\omega_1 + A_{22}\omega_2 = -\omega_1 + 2\omega_2 \ .$$

Let $u_i = \frac{1}{\sqrt{3}}e_i$, $i = 1,2$, be a rescaled version of the standard basis of $\mathbb{R}^2$. There is an isometry $E \to \mathbb{R}^2$ which maps

$$\alpha_1 \mapsto u_1 \quad , \quad \alpha_2 \mapsto -\tfrac{1}{2}u_1 + \tfrac{\sqrt{3}}{2}u_2 \ ,$$

and so also

$$\omega_1 \mapsto \tfrac{1}{2}u_1 + \tfrac{1}{2\sqrt{3}}u_2 \quad , \quad \omega_2 \mapsto \tfrac{1}{\sqrt{3}}u_2 \ .$$

The weight and root lattices are

$$\begin{aligned} \Lambda &= \{m_1\omega_1 + m_2\omega_2 \,|\, m_1,m_2 \in \mathbb{Z}\} \ , \\ \Lambda_r &= \{(2m_1 - m_2)\omega_1 + (2m_2 - m_1)\omega_2 \,|\, m_1,m_2 \in \mathbb{Z}\} \ . \end{aligned}$$

(How does one arrive at the expression for the root lattice?) ?

Lemma 6.1.3. 1. The Weyl group maps integral weights to integral weights.

2. For all $\lambda \in \Lambda$ there is $w \in W$ such that $w(\lambda)$ is dominant.

3. For each $\lambda \in \Lambda^+$, the number of dominant weights in $\lambda - \Lambda_r^+$ is finite.

Proof. Part 1: By Theorem 5.6.10, for all $w \in W$ and $\alpha \in \Phi$ also $w\alpha \in \Phi$. (How does that imply the statement?) ?

Part 2: By Theorem 5.6.15, there is a $w \in W$ such that $(w(\lambda), \alpha_i) \geq 0$ for $i = 1, \dots, r$.

Part 3: Let $\mu \in \lambda - \Lambda_r^+$ be dominant. By definition, this means that

- $(\mu, \alpha_i^\vee) \geq 0$ for $i = 1, \dots, r$, and
- $\lambda - \mu = \sum_{i=1}^r m_i \alpha_i$ for some $m_i \in \mathbb{Z}_{\geq 0}$.

Using this, we compute

$$\underbrace{(\lambda + \mu, \lambda - \mu)}_{=|\lambda|^2 - |\mu|^2} = \sum_{i=1}^r m_i \big((\lambda, \alpha_i) + (\mu, \alpha_i) \big) \geq 0$$

The right hand side is ≥ 0 as $m_i \geq 0$ by construction, and as λ, μ are dominant. Thus $|\mu| \leq |\lambda|$. But the intersection of the ball $B_{\leq |\lambda|} \subset E$ of radius $|\lambda|$ around the origin with Λ (and hence with Λ^+) is finite. $\square$

6.2 Highest weight representations

■ The next definition generalises the eigenspaces $M[\lambda]$ of h we defined for $sl(2, \mathbb{C})$-modules in Remark 5.5.3 to L-modules for all fssc L.

Definition 6.2.1. Let M be an L-module.

1. A $v \in M$ is a *vector of weight* $\lambda \in H^*$ if $h.v = \lambda(h)\, v$ for all $h \in H$. The *weight space for* λ is the corresponding eigenspace,

   $$M[\lambda] := \{ v \in M \,|\, \forall h \in H \, : \, h.v = \lambda(h)\, v \} \ .$$

   If $M[\lambda] \neq \{0\}$, then λ is called a *weight of* M.

2. The set of weights of M is denoted by

   $$P(M) := \{ \lambda \in H^* \,|\, M[\lambda] \neq \{0\} \} \ .$$

   We say that M *admits a weight decomposition* if

   $$M = \bigoplus_{\lambda \in P(M)} M[\lambda]$$

   as a $\mathbb{C}$-vector space (but not in general as an L-module).

Remark 6.2.2. 1. Let M be an L-module. If $v \in M[\lambda]$ and $x \in L_\alpha$, then $x.v \in M[\lambda + \alpha]$ as, for all $h \in H$,

$$h.(x.v) = [h, x].v + x.(h.v) = \alpha(h)x.v + \lambda(h)x.v = (\lambda + \alpha)(h)\, x.v \ .$$

2. The universal enveloping algebra $U(L)$ of $L \neq \{0\}$ is an example of an L-module, with $U(L)$ acting by multiplication (Why does this define a module?). However, M does not admit a weight decomposition. In fact, we have $U(L)[\lambda] = \{0\}$ for all $\lambda \in H^*$, so that $P(U(L)) = \emptyset$. (Why? *Hint:* For a given $h \in H$ consider a PBW-basis build from an ordered basis of L containing h as its smallest element. What can you say about h-eigenvectors?) ? ?

Example 6.2.3. $sl(2, \mathbb{C})$: Recall our standard basis e, h, f for $sl(2, \mathbb{C})$ from used above Proposition 2.1.9. There is one simple root α_1 (as $sl(2, \mathbb{C}$ has rank 1), and by the choice of base in Example 5.6.3 it is given by $\alpha_1 = \varepsilon_1 - \varepsilon_2$. In Example 6.1.2 we found the corresponding fundamental dominant weight to be $\omega_1 = \frac{1}{2}\alpha_1$. Thus

$$\alpha_1(h) = h_{11} - h_{22} = 2 \quad \text{and} \quad \omega_1(h) = \tfrac{1}{2}\alpha_1(h) = 1 \ .$$

1. Recall the Verma module V_λ ($\lambda \in \mathbb{C}$) from Exercise 25. This has a basis $\{v_m\}_{m \in \mathbb{Z}_{\geq 0}}$ on which $h \in sl(2, \mathbb{C})$ acts as $h.v_m = (\lambda - 2m)v_m = (\lambda - 2m)\omega_1(h)v_m$. Thus V_λ admits a weight decomposition, and the set of weights is
$$P(V_\lambda) = \{(\lambda - 2m)\omega_1 \,|\, m \in \mathbb{Z}_{\geq 0}\} \ .$$

2. Also in Exercise 25 we classified the irreducible finite-dimensional $sl(2, \mathbb{C})$ modules M_λ in terms of an integer $\lambda \in \mathbb{Z}_{\geq 0}$. The corresponding set of weights is finite,

$$P(M_\lambda) = \{\lambda\omega_1, (\lambda - 2)\omega_1, \ldots, (-\lambda + 2)\omega_1, -\lambda\omega_1\} \ .$$

Note that there is a 1-to-1 correspondence between dominant weights, i.e. elements of $\Lambda^+ = \mathbb{Z}_{\geq 0}\, \omega_1$, and irreducible finite-dimensional $sl(2, \mathbb{C})$-modules. This observation will generalise to all fssc L (Theorem 6.4.1 below).

■ Set $N_\pm := \bigoplus_{\alpha \in \Phi^\pm} L_\alpha \subset L$, so that

$$L \;=\; N_- \oplus H \oplus N_+ \ .$$

We define

$$B \;:=\; H \oplus N_+ \ .$$

$N_\pm$, H, and B are Lie subalgebras (but not ideals) of L (Why?). B is called a *Borel subalgebra*. ?

■ Define the *height of* $\lambda \in \Lambda_r$ *(with respect to* Δ*)* to be

$$\mathrm{ht}(\lambda) = \sum_{i=1}^{r} m_i \quad \text{for} \quad \alpha = \sum_{i=1}^{r} m_i \, \alpha_i$$

Note that the m_i are integers, and so also $\mathrm{ht}(\alpha) \in \mathbb{Z}$.

Lemma 6.2.4. 1. $N_\pm$ are nilpotent and B is solvable.

2. If M is a 1-dimensional B-module, then $M = M[\lambda]$ for some $\lambda \in H^*$ and $N_+ \subset B$ acts as zero on M.

Proof. Part 1: Recall from Definitions 3.1.1 and 3.2.1:

- *Descending central series:* $C^0L = L$, $C^{i+1}L = [L, C^iL]$; a Lie algebra is nilpotent if $C^nL = \{0\}$ for some n.
- *Derived series:* $D^0L = L$, $D^{i+1}L = [D^iL, D^iL]$; a Lie algebra is solvable if $C^nL = \{0\}$ for some n.

It follows from $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$, that

$$C^n N_+ \subset \bigoplus_{\beta_1,\dots,\beta_n \in \Phi^+} L_{\beta_1+\dots+\beta_n} \qquad (*)$$

Let $R = \max\{\mathrm{ht}(\alpha) | \alpha \in \Phi\}$ the maximal height among all roots. For $\beta \in \Phi^+$ we have $\mathrm{ht}(\beta) \geq 1$ (Why?). Hence for $\beta_1, \dots, \beta_n \in \Phi^+$ we get $\mathrm{ht}(\beta_1 + \dots + \beta_n) \geq n$. If we are summing more than R positive roots, the resulting height is strictly greater than R and hence cannot be a root. Combining this with $(*)$ shows ?

$$C^{R+1} N_+ = \{0\} \ .$$

The argument for N_- is analogous. For B note that $[B, B] \subset N^+$, and so nilpotency of N_+ implies that B is solvable.

Part 2: Let $v \in M$ be non-zero. Since M is one-dimensional, for each $h \in H$, $h.v$ is proportional to v. Hence there is a $\lambda \in H^*$ such that $h.v = \lambda(h)v$ for all $h \in H$. It follows that $M = M[\lambda]$.

Let $x_\alpha \in L_\alpha$ for some $\alpha \in \Phi^+$. Recall from Remark 6.2.2 (1) that $x_\alpha.v \in M[\lambda + \alpha]$. But since $M = M[\lambda]$, we must have $M[\lambda + \alpha] = \{0\}$. Thus $x_\alpha.v = 0$. As the x_α span N_+, it follows that $N_+.v = \{0\}$. □ (19.1.)

■ Let X be an arbitrary Lie algebra over some field k, and let M be an X-module. We say that M *is generated by a $v \in M$* if

$$U(X).v \;=\; M \;.$$

In other words, M is spanned by the vectors obtained from v by repeated action with elements from X.

Definition 6.2.5. A *highest weight module of weight λ (of L with respect to Δ)* is a non-zero L-module M such that

1. M is generated by some $v \in M[\lambda]$, and
2. $N_+.v = \{0\}$.

■ Note that whether an L-module is a highest weight representation or not depends on the choice of base Δ. (Can you given an example? *Hint:* Think of choices of base for $sl(2, \mathbb{C})$ and of the Verma module in Exercise 25.) ?

Remark 6.2.6. Every highest weight representation admits a weight decomposition (Exercise). E

(Ex. 42)

Proposition 6.2.7. Every irreducible finite-dimensional L-module is a highest weight representation.

Proof. Let M be an irreducible finite-dimensional L-module. Then M is in particular a B-module. Since B is solvable (Lemma 6.2.4), by Lie's theorem (Theorem 3.2.3) and Proposition 3.2.4 (1), it stabilises a flag (M_i) in M. In particular, M_1 is a 1-dimensional B-module. By Lemma 6.2.4, $M_1 = M_1[\lambda]$ for some $\lambda \in H^*$. In particular, $M_1 \subset M[\lambda]$.

Pick $v \in M_1$, $v \neq 0$. Again by Lemma 6.2.4. we have $N_+.v = \{0\}$. Set $M' = U(L).v \subset M$. Then M' is a non-zero L-submodule of M (Why?). But M is irreducible by assumption, and so we must have $M' = M$. ? □

6.3 Verma modules

■ For $\lambda \in H^*$ let C_λ be the 1-dimensional B-module with underlying vector space $\mathbb{C}$ and action

$$h.1 \;=\; \lambda(h) \quad , \quad x.1 \;=\; 0 \;\; (x \in N_+) \;.$$

Definition 6.3.1. The *Verma module of weight λ (of L with respect to H and Δ)* is

$$V_\lambda \;:=\; \mathrm{Ind}_B^L(\mathbb{C}_\lambda) \;\overset{\text{def.}}{=}\; U(L) \otimes_{U(B)} \mathbb{C}_\lambda \;.$$

Remark 6.3.2. *Claim:* V_λ is a highest weight representation of weight λ.

⌈ We verify Definition 6.2.5 for the choice $v = 1 \otimes_{U(B)} 1$. By definition of the relative tensor product (Definition 4.5.1) we have, for $h \in H$ and $x \in N_+$,

$$h.v = h \otimes_{U(B)} 1 = 1 \otimes_{U(B)} h.1 = \lambda(h)\, 1 \otimes_{U(B)} h.1 = \lambda(h)\, v \ ,$$
$$x.v = x \otimes_{U(B)} 1 = 1 \otimes_{U(B)} x.1 = 0 \ .$$

⌋

In particular, by Remark 6.2.6, V_λ admits a weight decomposition.

Lemma 6.3.3. For $\lambda \in H^*$ we have:

1. $P(V_\lambda) = \lambda - \Lambda_r^+$,
2. $\dim V_\lambda[\lambda] = 1$,
3. $\dim V_\lambda[\mu] < \infty$ for all $\mu \in H^*$.

Proof. We will use Corollary 4.5.7 to obtain a basis of $V_\lambda = \mathrm{Ind}_B^L(\mathbb{C}_\lambda)$. To this end, pick a total order on Φ^- and choose a non-zero vector $x_\alpha \in L_\alpha$ for each $\alpha \in \Phi^-$. Then a basis of B can be extended to a basis of L by adding $\{x_\alpha \,|\, \alpha \in \Phi^-\}$. Recall (from below Remark 4.3.2) that $S(\Phi^-)$ stands for the set of finite increasing sequences in Φ^-. A basis of V_λ is given by

$$\{v_M \,|\, M \in S(\Phi^-)\} \quad \text{where} \quad v_M := x_{\beta_1} \cdots x_{\beta_n} \otimes_{U(B)} 1 \text{ for } M = (\beta_1, \dots, \beta_n) \ .$$

For each $h \in H$ and $M = (\beta_1, \dots, \beta_n)$ we have

$$h.v_M \ = \ (\beta_1 + \cdots + \beta_n + \lambda)(h)\, v_M \ . \qquad (*)$$

(Details? *Hint:* Check that $h.v_{()} = \lambda(h)v_{()}$ and iterate the computation in Remark 6.2.2(1).) Thus the basis $\{v_M \,|\, M \in S(\Phi^-)\}$ already consists of common H-eigenvectors. ?

Part 1: By $(*)$, the set of weights of V_λ is given by

$$P(V_\lambda) = \{\lambda + \beta_1 + \cdots + \beta_n \,|\, n \in \mathbb{Z}_{\geq 0}, \beta_1, \dots, \beta_n \in \Phi^-\} \ .$$

But every such sum of β_i's can be written as $\sum_{i=1}^r m_i \alpha_i$ for $m_i \in \mathbb{Z}_{\leq 0}$ (this follows from property (B2) of a base). Conversely, as $-\alpha_i \in \Phi^-$, every such sum of simple roots occurs as a sum of β_i's. Using $\Phi^- = -\Phi^+$ gives $P(V_\lambda) = \lambda - \Lambda_r^+$.

Part 2: $\lambda + \beta_1 + \cdots + \beta_n = \lambda$ requires $\beta_1 + \cdots + \beta_n = 0$. As $\mathrm{ht}(\beta_i) \leq -1$ this in turn requires $n = 0$. Thus the only basis element in $(*)$ of weight λ is $v_{()}$.

Part 3: $V_\lambda[\mu]$ is non-zero only for $\mu \in P(V_\lambda) = \lambda - \Lambda_r^+$. Write $\mu = \lambda + \gamma$, $-\gamma \in \Lambda_r^+$. The basis vector $v_{(\beta_1,\dots,\beta_n)}$ lies in $V_\lambda[\mu]$ iff $\beta_1 + \dots + \beta_n = \gamma$. This in turn implies $\mathrm{ht}(\gamma) = \mathrm{ht}(\beta_1) + \dots + \mathrm{ht}(\beta_n)$. Since $\mathrm{ht}(\beta) \le -1$ for each $\beta \in \Phi^-$, there are only finitely many ways to satisfy this condition. $\square$

Lemma 6.3.4. Let M be a highest weight representation of weight λ. Then

1. $M \cong V_\lambda/W$ as L-modules for some L-submodule $W \subset V_\lambda$.
2. $\dim M[\lambda] = 1$ and $\dim M[\mu] < \infty$ for all $\mu \in H^*$.
3. $P(M) \subset P(V_\lambda)$

Proof. Part 1: We will use the universal property for induced modules from Exercise 23: Let $j : \mathbb{C}_\lambda \to V_\lambda = \mathrm{Ind}_B^L(\mathbb{C}_\lambda)$ be the B-module homomorphism which sends 1 to $v_{()}$. Then for all B-module homomorphisms $f : \mathbb{C}_\lambda \to M$, there exists a unique L-module homomorphism $\tilde{f} : \mathrm{Ind}_B^L(\mathbb{C}_\lambda) \to M$ such that $f = \tilde{f} \circ j$,

$$\begin{tikzcd} \mathbb{C}_\lambda \arrow[r, "j"] \arrow[d, "f"'] & \mathrm{Ind}_B^L(\mathbb{C}_\lambda) \arrow[dl, dashed, "\tilde{f}"] \\ M & \end{tikzcd}$$

Let $v \neq 0$ be a vector which exhibits M as a highest weight module of weight λ as in Definition 6.2.5. We choose f to map $1 \in \mathbb{C}_\lambda$ to v (Why is this a B-module homomorphism?). By construction, v is in the image of $\tilde{f}$, and as M is generated by v, $\tilde{f}$ is automatically surjective (Details?). Thus $M \cong \mathrm{Ind}_B^L(\mathbb{C}_\lambda)/W$ for $S := \ker(\tilde{f})$.

Parts 2 and 3: By Exercise 16, W is a direct sum of H-eigenspaces. Thus for all $\mu \in H^*$, $W[\mu] \subset V_\lambda[\mu]$. We can compute the quotient V_λ/W for each eigenspace separately,

$$M = \bigoplus_{\mu \in P(M)} M[\mu] \quad , \quad M[\mu] = V_\lambda[\mu]/W[\mu] \ .$$

In particular, $P(M) \subset P(V_\lambda)$. Since M is generated by a non-zero $v \in M[\lambda]$, we have $\dim M[\lambda] \ge 1$. On the other hand $\dim M[\lambda] \le \dim V_\lambda[\lambda] = 1$. Finally, $\dim M[\mu] \le \dim V_\lambda[\mu] < \infty$ by Lemma 6.3.3. $\square$

Proposition 6.3.5. For each $\lambda \in H^*$, the Verma module V_λ has a unique maximal proper submodule.

■ In other words, there exists a unique submodule $S_{\max} \subsetneq V_\lambda$ with the property that for all $S \subsetneq V_\lambda$ we have $S \subset S_{\max}$.

Proof of Proposition 6.3.5. Uniqueness is clear (Details?). For existence, define ?

$$S_{\max} = \mathrm{span}_{\mathbb{C}}\{\, W \mid W \subset V_\lambda \text{ submodule with } W[\lambda] = \{0\} \,\} \ .$$

Then:

- $S_{\max} \subset V_\lambda$ is a submodule. (Why?) ?
- For each submodule $S \subsetneq V_\lambda$ we have $S \subset S_{\max}$:
  By Exercise 16, S has a weight decomposition with $S[\mu] \subset V_\lambda[\mu]$. In particular, $S[\lambda] \subset V_\lambda[\lambda] = \mathbb{C}v_{()}$. Thus $S[\lambda] = \{0\}$ or else $v_{()} \in S[\lambda]$ and as v generates V_λ we would have $S = V_\lambda$. Altogether, $S[\lambda] = \{0\}$ and so $S \subset S_{\max}$ by construction.
- $S_{\max} \subsetneq V_\lambda$: As in the previous point, W which makes up the spanning set of $S_{\max}$ has a weight decomposition $W = \bigoplus_{\mu \in P(W)} W[\mu]$. By definition, $W[\lambda] = \{0\}$, so that $\lambda \notin P(W)$. It follows that each W in the spanning set of $S_{\max}$ is contained in the subspace
  $$W \subset \bigoplus_{\mu \in P(\lambda)\setminus\{\lambda\}} V_\lambda[\mu] \ .$$
  Hence also the linear span of such W is contained in that subspace. Altogether, $S_{\max}[\lambda] = \{0\}$. □

Remark 6.3.6. As in the above proof, write $S_{\max} \subsetneq V_\lambda$ for the maximal proper submodule and denote the quotient by

$$M_\lambda := V_\lambda / S_{\max} \ .$$

By construction,

- M_λ is irreducible,
- M_λ is a highest weight module,
- $\dim M_\lambda[\lambda] = 1$ and $\dim M_\lambda[\mu] < \infty$ for all $\mu \in H^*$,
- $P(M_\lambda) \subset P(V_\lambda) = \lambda - \Lambda_r^+$.

(The last two points also follow from Lemma 6.3.4, but why do the first two points hold?) ?

Lemma 6.3.7. Let $\lambda, \mu \in H^*$. The following are equivalent:

1. $\lambda = \mu$
2. $V_\lambda \cong V_\mu$ as L-modules
3. $M_\lambda \cong M_\mu$ as L-modules

(The proof is an exercise.) E

(Ex. 45)

 (23.1.)

6.4 Finite dimensional irreducible representations

The next theorem makes precise the statement that finite-dimensional irreducible L-modules are in 1-to-1 correspondence with weights in Λ^+.

Theorem 6.4.1. 1. Let $\lambda \in H^*$. M_λ is finite-dimensional iff $\lambda \in \Lambda^+$.

2. For every finite-dimensional irreducible L-module M there exists a unique $\lambda \in H^*$ such that $M \cong M_\lambda$ as L-modules.

We will prove this theorem in two stages: First part 2 and the direction $\Rightarrow$ in part 1, and then after some preparation we show the more involved direction $\Leftarrow$ in part 1.

Proof of Theorem 6.4.1. Part 2: By Proposition 6.2.7, M is a highest weight representation of weight λ for some $\lambda \in H^*$. By Lemma 6.3.4, $M \cong M_\lambda$, and by Lemma 6.3.7, the value of λ is unique.

Part 1, direction $\Rightarrow$: Suppose that M_λ is finite-dimensional. We need to show that then $(\lambda, \alpha_i^\vee) \in \mathbb{Z}_{\geq 0}$ for $i = 1, \dots, r$. Let $v \in M_\lambda[\lambda]$, $v \neq 0$ (this exists and is in fact a basis of $M_\lambda[\lambda]$ by Remark 6.3.6).

Recall from Proposition 5.5.2 the Lie subalgebra sl_{α_i} with basis $e_{\alpha_i} \in L_{\alpha_i}$, $f_{\alpha_i} \in L_{-\alpha_i}$ and $h_{\alpha_i} = H^{\alpha_i^\vee}$. We have

- M_λ is a finite-dimensional sl_{α_i}-module.
- $e_{\alpha_i} \in N_+$, so that $e_{\alpha_i}.v = 0$ by construction of M_λ.

The first point implies that $h_{\alpha_i}.v = mv$ for some $m \in \mathbb{Z}$ (Remark 5.5.3 (1)). Decomposing into irreducible sl_{α_i}-modules and comparing to the explicit action of e_{α_i} in Exercise 25 shows that the second point above implies $m \geq 0$. Altogether we have, for some $m \in \mathbb{Z}_{\geq 0}$,

$$mv = h_{\alpha_i}.v = \lambda(h_{\alpha_i}) = \lambda(H^{\alpha_i^\vee}) = (\lambda, \alpha_i^\vee) \ .$$

□

Lemma 6.4.2. Let N be a finite-dimensional $sl(2, \mathbb{C})$-module and $x \in N$, $x \neq 0$ such that $h.x = mx$. Then $m \in \mathbb{Z}$ and we have:

- if $m > 0$, then $f^m.x \neq 0$, and
- if $m < 0$, then $e^{-m}.x \neq 0$.

(Why does this hold? *Hint:* Decomposing N into irreducible $sl(2, \mathbb{C})$-modules and then verify that statement from the explicit e and f action in the classification of such modules in Exercise 25.) ?

Lemma 6.4.3. In $U(L)$ we have for $m \geq 0$ and $i, j = 1, \dots, r$:

$$\big[e_{\alpha_i}\,,\, f_{\alpha_j}^{m+1}\big] \;=\; \begin{cases} (m+1) f_{\alpha_j}^{m}\,(h_{\alpha_j} - m\mathbb{1}) & ; i = j \\ 0 & ; i \neq j \end{cases} \quad ,$$

where $\mathbb{1} \in U(L)$ denotes the unit.

Proof. Abbreviate $e = e_{\alpha_i}$ and $f = f_{\alpha_j}$. As the Lie bracket is a derivation, we have

$$[e, f^{m+1}] \;=\; [e, f^m] f + f^m [e, f] \ .$$

Both cases in the statement now follow inductively.

- $i = j$: $[e, f] = h$ gives the base case $m = 0$ (by convention anything in $U(L)$ to the power 0 is $\mathbb{1}$), as well as the inductive step. (Details?) ?
- $i \neq j$: $[e, f] = 0$ as $\alpha_i - \alpha_j$ is not a root (Property (B2) of a base). The above relation then immediately gives $[e, f^m] = 0$ for all m.

$\square$

Proof of Theorem 6.4.1. Part 1, direction $\Leftarrow$*:* Fix $\lambda \in \Lambda^+$. We want to show that M_λ is finite-dimensional.

Claim 1: For each $i = 1, \dots, r$, M_λ contains a non-zero finite-dimensional sl_{α_i}-submodule.

⌈ Let $v \neq 0$ be in $M_\lambda[\lambda]$ and $m_i = (\lambda, \alpha_i^\vee)$. Since $\lambda \in \Lambda^+$, we have $m_i \in \mathbb{Z}_{\geq 0}$. Set

$$W = \mathrm{span}_{\mathbb{C}}\{ f_{\alpha_i}^k.v \,|\, k = 0, 1, \dots, m_i \} \ .$$

We will check that W is an sl_{α_i}-submodule. To this end, it is enough to check that

$$e_{\alpha_i}.v = 0 \quad \text{and} \quad f_{\alpha_i}^{m_i+1}.v = 0 \ .$$

(Why is that enough?) The first identity is clear as $\alpha_i \in \Phi^+$ and $N_+.v = \{0\}$. For the second identity we first show that ?

$$e_{\alpha_j} f_{\alpha_i}^{m_i+1}.v \;=\; 0 \quad \text{for all} \ \ j = 1, \dots, r \ . \qquad (*)$$

Indeed, using Lemma 6.4.3 and $e_{\alpha_j}.v = 0$ for all j we get,

$$\begin{aligned} &e_{\alpha_j} f_{\alpha_i}^{m_i+1}.v = [e_{\alpha_j}, f_{\alpha_i}^{m_i+1}].v = 0 \quad (i \neq j) \\ &e_{\alpha_i} f_{\alpha_i}^{m_i+1}.v = [e_{\alpha_i}, f_{\alpha_i}^{m_i+1}].v = -(m_i+1) f_{\alpha_j}^{m}\,(m_i\mathbb{1} - h_{\alpha_j}).v = 0 \ , \end{aligned}$$

where the last step follows from $h_{\alpha_j}.v = (\lambda, \alpha_j^\vee)\, v = m_i\, v$. This establishes the identity $(*)$.

Abbreviate $w := f_{\alpha_i}^{m_i+1}.v$. It remains to show that $w = 0$.

Suppose $w \neq 0$. Then the weight of w is $\mu := \lambda - (m_i + 1)\alpha_i$ (Why?), in particular $\mu \neq \lambda$. But $(*)$ implies that $N_+.w = \{0\}$, and the universal property of the induced module V_μ implies that there is a non-zero L-module homomorphism $\varphi : V_\mu \to M_\lambda$ (Details? Why is this not directly implied by Lemma 6.3.4?). As M_λ is irreducible, φ descends to a non-zero homomorphism $M_\mu \to M_\lambda$, which by Schur's Lemma (Theorem 2.2.8) is an equivalence. This is a contradiction as Lemma 6.3.7 now implies $\lambda = \mu$. Thus $w = 0$. ⌋

Claim 2: For each $i = 1, \ldots, r$, M_λ is the union of its finite-dimensional sl_{α_i}-submodules.

⌈ Let $N \subset M_\lambda$ be the sum of all finite dimensional sl_{α_i}-submodules of W. We will show $N = M_\lambda$.

This shows the claim, as N is also equal to the union of all finite dimensional sl_{α_i}-submodules. (Why? *Hint:* Write $x \in N$ as $x = w_1 + \cdots + w_n$, where each w_i is contained in a finite-dimensional sl_{α_i}-submodule. What can you say about finite (not necessarily direct) sums of finite-dimensional sl_{α_i}-submodules?)

Let $Q \subset M_\lambda$ be a finite-dimensional sl_{α_i}-submodule. Consider the sub-vector space

$$Q' := Q + L.Q \;\subset\; M_\lambda \;.$$

Then Q' is again a finite-dimensional sl_{α_i}-submodule. For example, to see that Q' is closed under the action of e_{α_i}, one computes, for $q \in Q$, $x \in L$,

$$e_{\alpha_i} x.q = \underbrace{[e_{\alpha_i}, x]}_{\in L}.q + x\, \underbrace{e_{\alpha_i}.q}_{\in Q} \;\in\; Q' \;.$$

Since Q' is finite-dimensional, it follows that $Q' \subset N$. Thus N is not just closed under the action of sl_{α_i} but in fact under the action of all of L, i.e. it is an L-submodule of M_λ.

By Claim 1, N is non-zero. As M_λ is irreducible, we must have $N = M_\lambda$. ⌋

Claim 3: For each $w \in W$ and $\mu \in P(M_\lambda)$ also $w(\mu) \in P(M_\lambda)$.

⌈ By Theorem 5.6.15, it is enough to check the claim for simple reflections $w = s_{\alpha_i}$, $i = 1, \dots, r$.

Let $i \in 1, \dots, r$ be given. Pick $u \in M_\lambda[\mu]$, $u \neq 0$. We will use u to construct a non-zero element $\tilde{u} \in M_\lambda[s_{\alpha_i}(\mu)]$.

By Claim 2, $u \in Q$ for some finite-dimensional sl_{α_i}-submodule $Q \subset M_\lambda$. We have $h_{\alpha_i}.u = \mu(h_{\alpha_i})\,u = (\mu, \alpha_i^\vee)\,u$. Setting $m := (\mu, \alpha_i^\vee)$, from Lemma 6.4.2 we get $m \in \mathbb{Z}$ and

- if $m > 0$, then $\tilde{u} := f_{\alpha_i}^m.u \neq 0$, and
- if $m < 0$, then $\tilde{u} := e_{\alpha_i}^{-m}.u \neq 0$.

In both cases, $\tilde{u} \in M_\lambda[\mu - m\alpha_i]$. Finally,

$$\mu - m\alpha_i \;=\; \mu - (\mu, \alpha_i^\vee)\alpha_i \;=\; s_{\alpha_i}(\mu)\ .$$

⌋

(Claim 3 does not hold for Verma modules. Can you give an example?) ?

Claim 4: $P(M_\lambda)$ is finite.

⌈ Recall that by assumption $\lambda \in \Lambda^+$. Define $D := (\lambda - \Lambda_r^+) \cap \Lambda^+$. By Lemma 6.1.3, D is a finite set.

Let $\mu \in P(M_\lambda)$ be arbitrary. By Remark 6.3.6, $P(M_\lambda) \subset \lambda - \Lambda_r^+ \subset \Lambda$ (the second inclusion uses $\lambda \in \Lambda^+$ and $\Lambda_r^+ \subset \Lambda$). By Lemma 6.1.3, there is $w \in W$ such that $w(\mu)$ is dominant. By Claim 3, $w(\mu) \in P(M_\lambda)$, so that altogether $w(\mu) \in D$. We conclude that, conversely,

$$P(M_\lambda) \;\subset\; \{w(\nu) \,|\, \nu \in D, w \in W\}\ .$$

As both D and W are finite (recall Corollary 5.6.11), also $P(M_\lambda)$ is finite. ⌋

As $\dim(M_\lambda[\mu]) < \infty$ for all $\mu \in P(M_\lambda)$ (Remark 6.3.6), Claim 4 implies that M_λ is finite dimensional. □

The next lemma improves on Claim 3 in the above proof.

Corollary 6.4.4. For $\lambda \in \Lambda^+$ and $w \in W$ we have

$$\dim M_\lambda[\mu] \;=\; \dim M_\lambda[w(\mu)] \quad \text{for all} \quad \mu \in H^*\ .$$

(Why does this hold? *Hint:* From Theorem 6.4.1 we know M_λ is finite-dimensional, so in particular it is a finite dimensional sl_{α_i}-module. Taking the proof of Claim 3 as a guide, reduce the statement to the fact that for a finite-dimensional $sl(2,\mathbb{C})$-module (irreducible or not), the h-eigenspaces of eigenvalue m and $-m$ have the same dimension.) ?

Proposition 6.4.5. Let $\lambda \in \Lambda^+$. Then

$$P(M_\lambda) \;=\; \big\{ w(\mu) \,\big|\, w \in W, \mu \in (\lambda - \Lambda_r^+) \cap \Lambda^+ \big\} \;.$$

Proof. As in the proof of Claim 4 above, we use that for each $\mu \in \lambda$ there is a $w \in W$ such that $w(\mu) \in \Lambda^+$. Together with Corollary 6.4.4, the proposition follows if we can show:

$$(\lambda - \Lambda_r^+) \cap \Lambda^+ \;\subset\; P(M_\lambda) \;.$$

Let $\mu \in (\lambda - \Lambda_r^+) \cap \Lambda^+$ be given. We consider elements of the form $\nu = \mu + \sum_{i=1}^r k_i \alpha_i$ with $k_i \in \mathbb{Z}_{\geq 0}$. For an appropriate choice of the k_i we have $\nu = \lambda \in P(M_\lambda)$. The next claim shows that we can iteratively reduce the coefficients k_i by 1 without leaving the set $P(M_\lambda)$ until all k_i are zero. Hence the claim implies that $\mu \in P(M_\lambda)$.

Claim: Suppose that $\nu := \mu + \sum_{i=1}^r k_i \alpha_i \in P(M_\lambda)$ such that not all k_i are zero. Then there is an $i_0 \in \{1, \dots, r\}$ such that $k_{i_0} > 0$ and $\nu - \alpha_{i_0} \in P(M_\lambda)$.

⌈ Since $\nu \neq \mu$ we have $(\nu - \mu, \nu - \mu) > 0$. Thus there must be at least one i_0 such that $k_{i_0} > 0$ and $(\nu - \mu, \alpha_{i_0}) > 0$. From this we see that

$$(\nu, \alpha_{i_0}^\vee) \;>\; (\mu, \alpha_{i_0}^\vee) \;\in\; \mathbb{Z}_{\geq 0} \;, \qquad (*)$$

where the last inequality follows as $\mu \in \Lambda^+$.

By assumption, $\nu \in P(M_\lambda)$. Pick a non-zero $u \in M_\lambda[\nu]$. We compute

$$h_{\alpha_{i_0}}.u \;=\; \nu(h_{\alpha_{i_0}}).u \;=\; (\nu, \alpha_{i_0}^\vee).u \;.$$

Since M_λ is a finite dimensional (Theorem 6.4.1), we can use the explicit description of finite-dimensional $sl(2,\mathbb{C})$-modules to conclude that $(*)$ implies

$$f_{\alpha_{i_0}}.u \;\neq\; 0 \;.$$

The weight of $f_{\alpha_{i_0}}.u$ is $\nu - \alpha_{i_0}$. Thus $\nu - \alpha_{i_0} \in P(M_\lambda)$. ⌋

□

Example 6.4.6. Let us use Proposition 6.4.5 to compute the set of weights for some irreducible representations of $sl(2,\mathbb{C})$ and $sl(3,\mathbb{C})$:

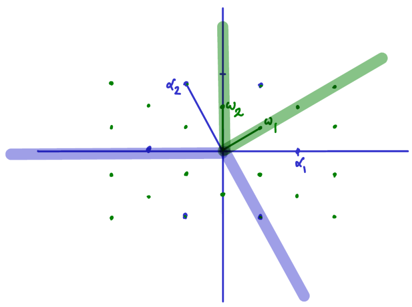
- $sl(2,\mathbb{C})$: Recall from Example 6.1.2 (1) that $\alpha_1 = 2\omega_1$. Let $\lambda = \Lambda^+$. Then $\lambda = m\omega_1$ for some $m \in \mathbb{Z}_{\geq 0}$ and

$$\begin{aligned}(\lambda - \Lambda_r^+) \cap \Lambda^+ &= \{m\omega_1 - k\alpha_1 \,|\, k \in \mathbb{Z}_{\geq 0}\} \cap \Lambda^+ \\ &= \{(m-2k)\omega_1 \,|\, k \in \mathbb{Z}_{\geq 0} \text{ and } 2k \leq m\} \ .\end{aligned}$$

The Weyl group is generated by simple reflections (Theorem 5.6.15), of which in this case there is only one. Thus $W \cong \mathbb{Z}_2$ with non-trivial element s_{α_1} acting by $s_{\alpha_1}(\mu) = -\mu$. Thus (26.1.)

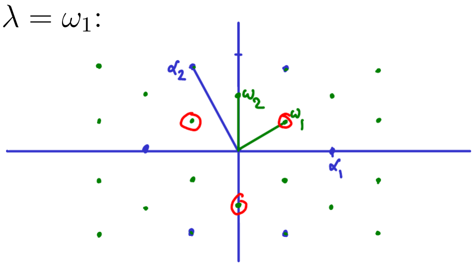
$$\begin{aligned}W.\big((\lambda - \Lambda_r^+) \cap \Lambda^+\big) &= \{\pm(m-2k)\omega_1 \,|\, k \in \mathbb{Z}_{\geq 0} \text{ and } 2k \leq m\} \\ &= P(M_\lambda) \ .\end{aligned}$$

- $sl(3,\mathbb{C})$: The weight lattice Λ (green) and the root lattice Λ_r (blue), together with the cones giving Λ^+ and $-\Lambda_r^+$:

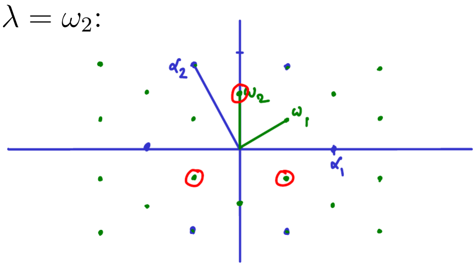


The Weyl group orbits of $(\lambda - \Lambda_r^+) \cap \Lambda^+$ for different choices of λ, giving $P(M_\lambda)$ for the corresponding irreducible $sl(3,\mathbb{C})$ representation M_λ.

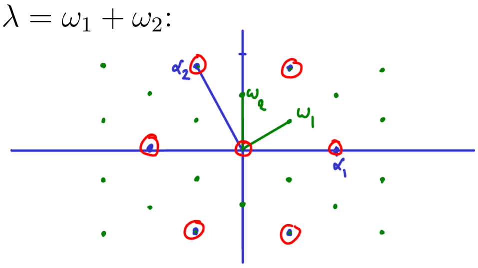
$\lambda = \omega_1$:



$\lambda = \omega_2$:



$\lambda = \omega_1 + \omega_2$:



■ Note that Proposition 6.4.5 just allows us to determine which $M_\lambda[\mu]$ are non-zero, and not their actual dimension. But in the three $sl(3,\mathbb{C})$ examples above, we can identify the representations explicitly. Combining Examples 5.6.3 and 6.1.2 we get

$$\omega_1 = \tfrac{1}{3}(2\alpha_1 + \alpha_2) = \tfrac{1}{3}(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3) = \varepsilon_1 \ , \quad \omega_2 = \tfrac{1}{3}(\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3) = -\varepsilon_3 \ ,$$

where we used $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$. Recall that the CSA consists of traceless diagonal matrices, which we write as

$$h \ = \ h_1 E_{11} + h_2 E_{22} + h_3 E_{33} \quad \text{with} \quad h_1 + h_2 + h_3 = 0 \ .$$

Then:

- *Fundamental representation:* Let F be the representation of $sl(3,\mathbb{C})$ with underlying vector space $\mathbb{C}^3$ and action by matrix multiplication. F is irreducible (Why?). The weight spaces of F are just $\mathbb{C}e_i$, $i = 1, 2, 3$, and we have ?

$$\begin{aligned} h.e_1 &= h_1 e_1 = \omega_1(h)\, e_1 \ , \\ h.e_2 &= h_2 e_2 = (-h_1 - h_3)e_2 = (-\omega_1 + \omega_2)(h)\, e_2 \ , \\ h.e_3 &= h_3 e_3 = -\omega_2(h)\, e_3 \ . \end{aligned}$$

  We conclude $F \cong M_{\omega_1}$. (Why is e_1 a highest weight vector? I.e. why does e_1 satisfy $N_+.e_1 = \{0\}$?) ?

- *Anti-fundamental representation:* Let $\overline{F}$ have again $\mathbb{C}^3$ as underlying vector space, but now $M \in sl(3,\mathbb{C})$ acts by matrix multiplication with $-M^t$. One can check that $\overline{F} \cong F^*$, see Definition 2.2.4 (4) (Details?), in particular $\overline{F}$ is irreducible. For $\overline{F}$ we find ?

$$h.e_1 = -\omega_1(h)e_1 \ , \quad h.e_2 = (\omega_1 - \omega_2)(h)e_2 \ , \quad h.e_3 = \omega_2(h)\, e_3 \ ,$$

  and so $F \cong M_{\omega_2}$. (Why is e_3 a highest weight vector?) ?

- *Adjoint representation:* The adjoint representation of $sl(3,\mathbb{C})$ is irreducible (Why?) is also irreducible, and the weight decomposition is ?

$$sl(3,\mathbb{C}) = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha \ ,$$

and one finds $(sl(3,\mathbb{C}),\mathrm{ad}) \cong M_{\omega_1+\omega_2}$. (Why? And what is the explicit matrix that is the corresponding highest weight vector?) ?

Note that for the fundamental and anti-fundamental representation, all weight spaces are 1-dimensional, while $\dim M_{\omega_1+\omega_2}[0] = 2$ for the adjoint representation.

Remark 6.4.7. 1. Combining Weyl's Theorem (Theorem 5.3.2) and Theorem 6.4.1, it follows that every finite-dimensional L-module M has a weight decomposition. One also finds that Corollary 6.4.4 holds for finite-dimensional modules, irrespective of irreducibility: for all $w \in W$ and $\mu \in H^*$ we have (Why?) ?

$$\dim M[\mu] \;=\; \dim M[w(\mu)] \ .$$

2. Theorem 6.4.1 can be stated as follows. For a rank r fssc Lie algebra L, one can specify an irreducible finite-dimensional representation uniquely by elements $(\lambda_1,\dots,\lambda_r) \in (\mathbb{Z}_{\geq 0})^r$, where such an r-tuple corresponds to $\lambda = \sum_i \lambda_i\omega_i \in \Lambda^+$. The λ_i are sometimes called *Dynkin labels of an irreducible representation.* For the $sl(3,\mathbb{C})$-representations discussed above we have:

representation	fundamental	anti-fund.	adjoint
Dynkin labels	(1,0)	(0,1)	(1,1)

6.5 Characters

■ Let $\mathbb{C}[\Lambda]$ be the group algebra of the weight lattice Λ, thought of as an abelian group under addition. $\mathbb{C}[\Lambda]$ has:

- *basis:* $\{e^\lambda \,|\, \lambda \in \Lambda\}$, where e^λ at this point is just a symbol, and
- *multiplication:* $e^\lambda e^\mu = e^{\lambda+\mu}$.
- *unit:* $\mathbb{1} := e^0$.

We define the algebra automorphism $\overline{(-)}$ of $\mathbb{C}[\Lambda]$ on a basis as

$$\overline{(-)} : \mathbb{C}[\Lambda] \to \mathbb{C}[\Lambda] \quad , \quad e^\lambda \mapsto e^{-\lambda} \ .$$

Remark 6.5.1. We can think of e^λ as a (non-linear) function $H \to \mathbb{C}$ via $e^\lambda(h) := e^{\lambda(h)}$, where the exponential on the right hand side is that of complex numbers. This defines a linear map

$$\phi : \mathbb{C}[\Lambda] \to \mathrm{Fun}(H, \mathbb{C}) \quad , \quad e^\lambda \mapsto \big(h \mapsto e^{\lambda(h)}\big) \ .$$

This map is injective (Why? *Hint:* It is enough to show that the functions $h \to e^{\lambda(h)}$ are linearly independent. Write $\lambda = \sum_i m_i \omega_i$ and $t = \sum t_i \alpha_i^\vee$. Then $\lambda(h) = \sum m_i t_i$, and so it is enough to show that the functions $(t_1, \dots, t_r) \mapsto e^{m_1 t_1 + \dots + m_r t_r}$ are linearly independent for different $m \in \mathbb{Z}^r$). Furthermore, ϕ is an algebra homomorphism, where $\mathrm{Fun}(H, \mathbb{C})$ is an algebra via pointwise addition and multiplication (Details?). ?

Definition 6.5.2. The *character* of a finite-dimensional L-module M is

$$\mathrm{ch}(M) := \sum_{\mu \in P(M)} \dim\big(M[\mu]\big)\, e^\lambda \ \in \ \mathbb{C}[\Lambda] \ .$$

Remark 6.5.3.

1. The character is well-defined as both $P(M)$ and $\dim M[\mu]$ are finite for a finite-dimensional L-module M.

2. Applying ϕ from Remark 6.5.1 to $\mathrm{ch}(M)$ results in a function $H \to \mathbb{C}$. This function can be described by a trace as follows. Let $R : L \to \mathrm{End}(M)$ be the Lie algebra homomorphism describing M as a representation. Then $R(h)$ is a linear map $M \to M$. As M is finite-dimensional, we can consider its matrix exponential $e^{R(h)} : M \to M$, which is again a linear map (Section 1.5).

   Claim: We have, for $h \in H$,

   $$\phi\big(\,\mathrm{ch}(M)\big)(h) \ = \ \mathrm{tr}_M\big(e^{R(h)}\big) \ .$$

   ⌈ By Remark 6.4.7, M has a weight decomposition $M = \bigoplus_{\mu \in P(M)} M[\mu]$. The weight spaces are eigenspaces for the action of h: for $x \in M[\mu]$ we have $R(h)x = \lambda(h)x$. Hence the matrix exponential on $M[\mu]$ is just $e^{R(h)}x = e^{\lambda(h)}x$. Using this, we compute

   $$\mathrm{tr}_M\big(e^{R(h)}\big) \ = \sum_{\mu \in P(M)} \mathrm{tr}_{M[\mu]}\big(e^{R(h)}\big) \ = \sum_{\mu \in P(M)} \mathrm{tr}_{M[\mu]}\big(e^{\lambda(h)}\,\mathrm{id}\,\big)$$
   $$= \sum_{\mu \in P(M)} e^{\lambda(h)} \dim M[\mu] \ = \ \phi\big(\,\mathrm{ch}(M)\big)(h)$$

   ⌋

Example 6.5.4. 1. $sl(2,\mathbb{C})$: For $\lambda = m\omega_1$ we have $P(M_\lambda) = \{(m-2k)\omega_1 \,|\, k = 0,1,\dots,m\}$. Thus

$$\mathrm{ch}(M_\lambda) = e^{m\omega_1} + e^{(m-2)\omega_1} + \dots + e^{-m\omega_1} \ .$$

If we abbreviate $x := e^{\omega_1}$, we have $e^{m\omega_1} = x^m$ and we can rewrite the character as

$$\mathrm{ch}(M_\lambda) = \frac{x^{m+1} - x^{-m-1}}{x - x^{-1}} \ .$$

2. $sl(3,\mathbb{C})$: Abbreviate $x = e^{\omega_1}$ and $y = e^{\omega_2}$. Using the explicit realisations in Example 6.4.6 we find (Details?) ?

$$\begin{aligned} \mathrm{ch}(M_{\omega_1}) &= x + x^{-1}y + y^{-1} \ , \\ \mathrm{ch}(M_{\omega_2}) &= y + xy^{-1} + x^{-1} \ , \\ \mathrm{ch}(M_{\omega_1+\omega_2}) &= xy + x^{-1}y^2 + x^2y^{-1} + 2 + xy^{-2} + x^{-2}y + x^{-1}y^{-1} \ . \end{aligned}$$

Theorem 6.5.5. Let M, N be finite-dimensional L-modules. Then $M \cong N$ as L-modules iff $\mathrm{ch}(M) = \mathrm{ch}(N)$.

The proof uses the next two lemmas.

Lemma 6.5.6. The set $\{\mathrm{ch}(M_\lambda) \,|\, \lambda \in \Lambda^+\}$ is a linearly independent subset of $\mathbb{C}[\Lambda]$.

Proof. Define a partial order on Λ via

$$\mu \le \lambda \overset{\text{Def.}}{\Leftrightarrow} \lambda - \mu \in \Lambda_r^+ \ .$$

(Why is this a partial order? *Hint:* One needs to check $\lambda \le \lambda$; $\mu \le \lambda$ and $\lambda \le \mu$ implies $\lambda = \mu$; $\nu \le \mu$ and $\mu \le \lambda$ implies $\nu \le \lambda$.) Let $S \subset \Lambda^+$ be a finite subset and suppose we have ?

$$\sum_{\lambda \in S} c_\lambda \, \mathrm{ch}(M_\lambda) \ = \ 0 \quad \text{for some} \quad c_\lambda \in \mathbb{C} \ .$$

We need to show that all c_λ are zero.

Pick a $\lambda_0 \in S$ which is maximal with respect to the partial order "$\le$", i.e. $\lambda_0 \le \mu \Rightarrow \lambda_0 = \mu$ (as S is finite, this is always possible, thought λ_0 is typically not unique). Then

- e^{λ_0} is a summand of $\mathrm{ch}(M_{\lambda_0})$ as $\dim M_{\lambda_0}[\lambda_0] = 1$, and
- e^{λ_0} is not a summand of $\mathrm{ch}(M_\mu)$ for any $\mu \in S \setminus \{\lambda_0\}$.
(Why? *Hint:* $P(M_\mu) \subset \mu - \Lambda_r^+$) ?

As the e^λ are linearly independent by definition, it follows that $c_{\lambda_0} = 0$. Iterating this procedure shows that all c_λ are zero. $\square$

■ Recall from Corollary 5.3.7 that L has a unique (up to isomorphism) 1-dimensional representation, which we will denote simply by $\mathbb{C}$. Recall also the notation $\overline{e^\lambda} = e^{-\lambda}$.

Lemma 6.5.7. Let M, N be finite-dimensional L-modules. Then:

1. If $M \cong N$, then $\mathrm{ch}(M) = \mathrm{ch}(N)$.
2. $\mathrm{ch}(\mathbb{C}) = 1$.
3. $\mathrm{ch}(M \oplus N) = \mathrm{ch}(M) + \mathrm{ch}(N)$.
4. $\mathrm{ch}(M \otimes N) = \mathrm{ch}(M) \cdot \mathrm{ch}(N)$.
5. $\mathrm{ch}(M^*) = \overline{\mathrm{ch}(M)}$.

Proof. 1. Clear. (Details?) ?

2. Clear.
3. We use the representation as a trace in Remark 6.5.3 (2) and the fact that $\phi : \mathbb{C}[\Lambda] \to \mathrm{Fun}(H, \mathbb{C})$ from Remark 6.5.1 is injective. For $m \oplus n \in M \oplus N$ we have $R_{M\oplus N}(h)(m \oplus n) = (R_M(h)m) \oplus (R_N(h)n)$, and so also
$$e^{R_{M\oplus N}(h)}(m \oplus n) = \left(e^{R_M(h)}m\right) \oplus \left(e^{R_N(h)}n\right)$$
For the trace, this implies
$$\mathrm{tr}_{M\oplus N}\left(e^{R_{M\oplus N}(h)}\right) = \mathrm{tr}_M\left(e^{R_M(h)}\right) + \mathrm{tr}_N\left(e^{R_N(h)}\right) .$$
4. The action of L on the tensor product $M \otimes N$ is defined as (Definition 2.3.2)
$$\begin{aligned} R_{M\otimes N}(h)(m \otimes n) &= (R_M(h)m) \otimes n + m \otimes (R_N(h)n) \\ &= \big(R_M(h) \otimes \mathrm{id}_N + \mathrm{id}_M \otimes R_N(h)\big)(m \otimes n) . \end{aligned}$$
The two linear endomorphisms $R_M(h) \otimes \mathrm{id}_N$ and $\mathrm{id}_M \otimes R_N(h)$ of $M \otimes N$ commute, and hence the matrix exponential can be expanded as (Remark 1.5.1)
$$\begin{aligned} e^{R_{M\otimes N}(h)}(m \otimes n) &= e^{R_M(h)\otimes \mathrm{id}_N} e^{\mathrm{id}_M \otimes R_N(h)}(m \otimes n) \\ &= e^{R_M(h)}m \otimes e^{R_N(h)}n \end{aligned}$$
Taking the trace shows the claim.

5. Analogously. (Details? *Hint:* Let $f : V \to V$ be a linear endomorphism of a finite-dimensional vector space V, and let $f^* : V^* \to V^*$ be its dual. Check that $\mathrm{tr}_V(f) = \mathrm{tr}_{V^*}(f^*)$ and that $\exp(f)^* = \exp(f^*) : V^* = V^*$. The action of L on M^* is given in Definition 2.2.4). ? $\square$ (30.1.)

Proof of Theorem 6.5.5. The direction "⇒" was shown in Lemma 6.5.7 (1). For the converse direction, suppose that $\mathrm{ch}(M) = \mathrm{ch}(N)$. By Weyl's Theorem (Theorem 5.3.2), M, N are direct sums of irreducible L-modules,

$$M \cong \bigoplus_{\lambda \in \Lambda^+} (M_\lambda)^{\oplus m_\lambda} \quad , \quad N \cong \bigoplus_{\lambda \in \Lambda^+} (M_\lambda)^{\oplus n_\lambda} \ ,$$

where $m_\lambda, n_\lambda \in \mathbb{Z}_{\geq 0}$ and only finitely many are nonzero. By parts 1 and 3 of Lemma 6.5.7, this implies

$$\mathrm{ch}(M) = \sum_{\lambda \in \Lambda^+} m_\lambda \, \mathrm{ch}(M_\lambda) \quad , \quad \mathrm{ch}(N) = \sum_{\lambda \in \Lambda^+} n_\lambda \, \mathrm{ch}(M_\lambda) \ .$$

By assumption we have $\mathrm{ch}(M) = \mathrm{ch}(N)$, and so linear independence of the $\mathrm{ch}(M_\lambda)$ (Lemma 6.5.6) implies $m_\lambda = n_\lambda$ for all λ, and hence $M \cong N$. $\square$

Example 6.5.8. 1. *Tensor products of $sl(2, \mathbb{C})$-modules:* In Exercise 9 we showed that $M_{\omega_1} \otimes M_{\omega_1} \cong M_0 \oplus M_{2\omega_1}$ by explicitly constructing an isomorphism of $sl(2, \mathbb{C})$-modules. Using Theorem 6.5.5, Lemma 6.5.7 and Example 6.5.4, we obtain a much shorter proof:

$$\begin{aligned} \mathrm{ch}(M_{\omega_1} \otimes M_{\omega_1}) &= \mathrm{ch}(M_{\omega_1})\,\mathrm{ch}(M_{\omega_1}) = (x + x^{-1})^2 \\ &= x^2 + 2 + x^{-2} = \mathrm{ch}(M_0) + \mathrm{ch}(M_{2\omega_1}) \ . \end{aligned}$$

In fact, we can compute the decomposition of a general tensor product $M_{m\omega_1} \otimes M_{n\omega_1}$ into irreducibles via

$$\begin{aligned} \mathrm{ch}(M_{m\omega_1})\,\mathrm{ch}(M_{n\omega_1}) &= \frac{x^{m+1} - x^{-m-1}}{x - x^{-1}} \cdot \frac{x^{n+1} - x^{-n-1}}{x - x^{-1}} \\ &= \frac{x^{m+1} - x^{-m-1}}{x - x^{-1}} \big(x^n + x^{n-2} + \cdots + x^{-n+2} + x^{-n} \big) \\ &= \sum_{k=|m-n|,\text{ step } 2}^{m+n} \mathrm{ch}(M_{k\omega_1}) \end{aligned}$$

(Details? *Hint:* The expansion in the second step is most useful if we assume without loss of generality that $m \geq n$.) ?

2. *Tensor products of $sl(3,\mathbb{C})$-modules:* Using the characters from Example 6.5.4, let us decompose $M_{\omega_1} \otimes M_{\omega_2}$ into irreducibles:

$$\begin{aligned}
\mathrm{ch}(M_{\omega_1})\,\mathrm{ch}(M_{\omega_2}) &= (x + x^{-1}y + y^{-1})(y + xy^{-1} + x^{-1}) \\
&= xy + x^{-1}y^2 + x^2y^{-1} + 3 + xy^{-2} + x^{-2}y + x^{-1}y^{-1} \\
&= \mathrm{ch}(M_0) + \mathrm{ch}(M_{\omega_1+\omega_2}) \ .
\end{aligned}$$

Thus $M_{\omega_1} \otimes M_{\omega_2} \cong M_0 \oplus M_{\omega_1+\omega_2}$.

■ Recall the notation $\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ from Corollary 5.6.14. For an element $w \in W$ define the *length $\ell(w)$ of w* to be the minimal number of simple reflections needed to obtain w. The determinant of $w : H^* \to H^*$ is related to the length of w via (Why?) ?

$$(-1)^{\ell(w)} = \det(w) \ .$$

The next theorem we will state without proof. (See Humphreys, Theorem 24.3 or Kirillov, Theorem 8.34 for a proof.)

Theorem 6.5.9 (Weyl's Character Formula)**.** For $\lambda \in \Lambda^+$ we have

$$\mathrm{ch}(M_\lambda) = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\delta)}}{\prod_{\alpha\in\Phi^+}\left(e^{\alpha/2} - e^{-\alpha/2}\right)} \ .$$

Example 6.5.10. 1. *$sl(2,\mathbb{C})$:* We have $\Phi^+ = \{\alpha_1\}$ and so $\delta = \frac{1}{2}\alpha_1 = \omega_1$ and

$$\prod_{\alpha\in\Phi^+}\left(e^{\alpha/2} - e^{-\alpha/2}\right) = e^{\alpha_1/2} - e^{-\alpha_1/2} = e^{\omega_1} - e^{-\omega_1} = x - x^{-1} \ .$$

The Weyl group acts as $W = \{\mathrm{id}, -\mathrm{id}\}$ on E, so that, for $\lambda = m\omega_1$,

$$\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\delta)} = e^{\lambda+\delta} - e^{-\lambda-\delta} = x^{m+1} - x^{-m-1} \ .$$

Combining this, the Weyl character formula gives

$$\mathrm{ch}(M_{m\omega_1}) = \frac{x^{m+1} - x^{-m-1}}{x - x^{-1}}$$

as in Example 6.5.4.

2. *$sl(3,\mathbb{C})$:* This example illustrates that the Weyl character formula quickly becomes unwieldy. Recall that $\alpha_1 = 2\omega_1 - \omega_2$ and $\alpha_2 =$

$-\omega_1 + 2\omega_2$ (cf. Example 6.1.2) We have $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ and so

$$\delta \;=\; \tfrac{1}{2}(\alpha_1 + \alpha_2 + (\alpha_1 + \alpha_2)) \;=\; \omega_1 + \omega_2 \ .$$

The denominator of the Weyl character formula is given by

$$\begin{aligned} D :&= \prod_{\alpha\in\Phi^+} \left(e^{\alpha/2} - e^{-\alpha/2}\right) \\ &= \left(xy^{-\frac12} - x^{-1}y^{\frac12}\right)\left(x^{-\frac12}y - x^{\frac12}y^{-1}\right)\left(x^{\frac12}y^{\frac12} - x^{-\frac12}y^{-\frac12}\right) \ . \end{aligned}$$

To compute the numerator, we need a more explicit handle on the Weyl group. The simple reflections act on the fundamental weights as

$$s_{\alpha_i}(\omega_j) = \omega_j - (\omega_j, \alpha_i^\vee)\alpha_i = \omega_j - \delta_{i,j}\alpha_i \ ,$$

and so

$$\begin{aligned} s_{\alpha_1}(a\omega_1 + b\omega_2) \;&=\; a\omega_1 - a\alpha_1 + b\omega_2 \;=\; -a\omega_1 + (a+b)\omega_2 \ , \\ s_{\alpha_2}(a\omega_1 + b\omega_2) \;&=\; a\omega_1 + b\omega_2 - b\alpha_2 \;=\; (a+b)\omega_1 - b\omega_2 \ . \end{aligned}$$

Abbreviating $a\omega_1+b\omega_2$ as (a,b) and using that W is generated by simple reflections (Theorem 5.6.15), it is now easy to compute the action of $w \in W$ on (a,b):

w	id	s_{α_1}	$s_{\alpha_2}s_{\alpha_1}$
$w.(a,b)$	(a,b)	$(-a, a+b)$	$(b, -a-b)$
$\det(w)$	1	-1	1

w	s_{α_2}	$s_{\alpha_1}s_{\alpha_2}$	$s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}$
$w.(a,b)$	$(a+b, -b)$	$(-a-b, a)$	$(-b, -a)$
$\det(w)$	-1	1	-1

As an example, we compute $\mathrm{ch}(M_{\omega_1})$. Then $\omega_1 + \delta = (2,1)$ and the W-orbit of $(2,1)$ is

$$W.(2.1) \;=\; \big\{(2,1)\,,\,(-2,3)\,,\,(1,-3)\,,\,(3,-1)\,,\,(-3,2)\,,\,(-1,-2)\big\} \ .$$

The gives the numerator as

$$\begin{aligned} N :&= \sum_{w\in W} (-1)^{\ell(w)} e^{w(\omega_1+\delta)} \\ &= x^2y - x^{-2}y^3 + xy^{-3} - x^3y^{-1} + x^{-3}y^2 - x^{-1}y^{-2} \ . \end{aligned}$$

One can now check that indeed as in Example 6.5.4 we have

$$\mathrm{ch}(M_{\omega_1}) \;=\; \frac{N}{D} \;=\; x + x^{-1}y + y^{-1} \ .$$

(Check! Can you repeat the computation for $\mathrm{ch}(M_{\omega_2})$ and $\mathrm{ch}(M_{\omega_1+\omega_2})$?) ?