Tensor Categories and Topological Field Theory

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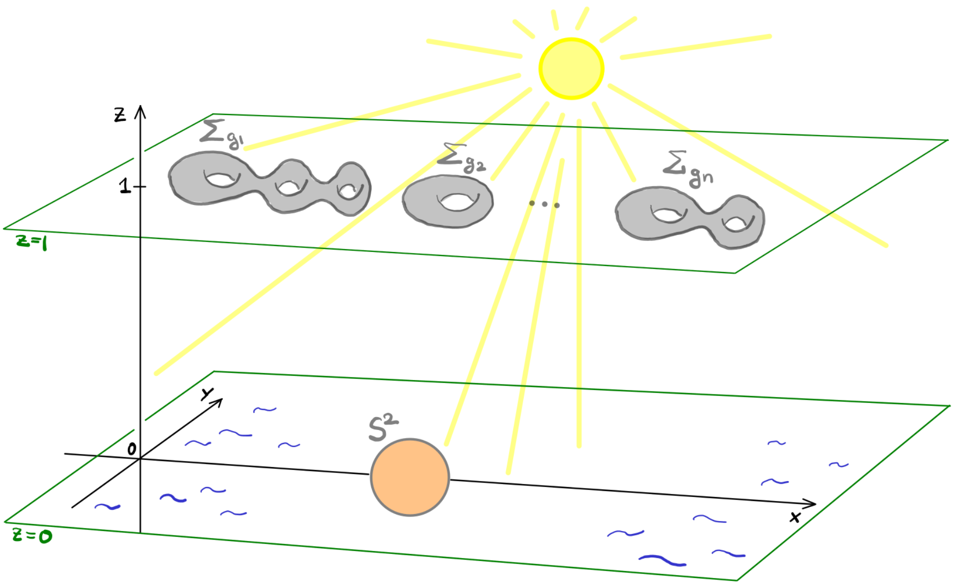

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About these notes

These are the notes of a lecture given in Hamburg in the summer term 2026 for Masters students in Mathematics and Mathematical Physics. The format was two lectures and one tutorial of 90 minutes a week each.

There are three design decisions which shape how the material is presented: the categorical background is developed as needed in the application to topological field theory; TFTs are discussed dimension by dimension from 1 to 3; the underlying bordism categories include stratified bordisms from the start to treat TFTs with defects.

There are advantages and disadvantages in this approach. On the one hand, the mathematical contents is a bit scattered across chapters. For example, braided categories are introduced in Section 2.3 (to say what "symmetric" means in the definition of a TFT), but ribbon categories only appear in Section 5.1 (when discussing 3d TFT). On the other hand, proceeding by dimension makes the complexity of the constructions increase gradually, which I think helps to get a feel for working with TFTs. And as different aspects of TFTs are in the foreground in each section, this approach is not repetitive.

Some topics covered in this course are less standard. For example, we discuss invertible n-dimensional TFTs and stacking of TFTs. The trivial n-dimensional TFT with line defects labelled by a symmetric monoidal category S is introduced and used as basic building block in the 2d state sum construction, and also as a trivial example in the 3d section. The state sum constructions in the 1d and 2d sections are presented as a special case of partition functions of non-topological lattice models. As a by-product, this gives a functorial description of lattice models in terms of defect TFT, though this aspect is not treated in any depths.

TFTs have a general symmetric monoidal target, both in the state sum constructions in the 1d and 2d case, and in the surgery and universal construction in the 3d case. That last part generalises the Reshetikhin-Turaev construction from modular fusion categories to twist-nondegenerate ribbon fusion categories (with the TFT taking values in the symmetric centre), and the approach via the universal construction used here is new to the best of my knowledge.

Hopf algebras are only mentioned in side remarks in these notes. That is mostly due to lack of time, but also opens some space to discuss braided fusion categories in terms of F- and R-matrices. This is the general description of the algebraic input for the (semisimple) 3d surgery construction, while Hopf algebras provide only special examples. The Fibonnacci and Ising fusion categories are described in some detail - both are not representation categories of Hopf algebras.

I should point out that "tensor categories" in the technical sense used in [EGNO] actually also do not appear in these notes, other than in their semisimple incarnation as fusion categories. In the title the term is meant more colloquially as "monoidal cateogries, possibly with some extra structure".

The notes end with a much-too-terse treatment of surface mapping class groups. It would have been good to discuss this and other applications of the surgery 3d TFTs developed with much effort, but at that point we had reached the end of the term.

I would like to thank the participants of the course for taking part with interest and for asking many pertinent questions. Many thanks for comments on the script go to Max-Niklas Steffen, Julian Stolz, and Alexander Tishchenko. I thank Yilong Wang for discussions on the WRT invariants, and Nils Carqueville and Vincentas Mulevičius for helpful input on PL-bordisms and defects.

This script was entirely conceived and written by a human (me), proofread for mathematical errors and typos by an AI (Claude, Sonnet 5), and the suggested corrections were then checked and implemented by me.

1 Introduction

Here is the first main player of this course:[1]

Definition 1.1. An *n-dimensional topological field theory (TFT)* is a symmetric monoidal functor from n-dimensional bordisms to vector spaces.

The second main player, tensor categories, will be introduced when we explain the words in the above definition in the next section.

In this course, we want to understand TFTs as a bridge between algebra and topology,

$$\big(\text{algebraic objects}\big) \quad \longleftrightarrow \quad \big(\text{topological/geometric objects}\big) \ ,$$

and we will see that each side has interesting things to say about the other. But the original motivation behind this definition comes from the (non-rigorous) notion of path integrals in quantum field theory, which we briefly recall next.

Motivation from path integrals

Very roughly, one describes the state Φ of a quantum system as a complex valued function on a space of "classical configurations". The time evolution of a state Φ at time $t = t_0$ to a state Φ' at time $t = t_1$ is described by "averaging over all paths in configuration space", assigning a complex weight to each path. This is the idea of the path integral.

Let us apply this to n-dimensional quantum field theory.[2] The space of classical configurations is the space of functions $\mathrm{Fun}(N, T)$ from an $n-1$ dimensional manifold N to some fixed space T. For example $T = \mathbb{R}$ for a real-valued scalar field. The space of states $\mathcal{H}(N)$ on N consists of complex valued maps on $\mathrm{Fun}(N, T)$,

$$\mathcal{H}(N) = \mathrm{Fun}\big(\mathrm{Fun}(N, T)\,,\, \mathbb{C}\big) \ .$$

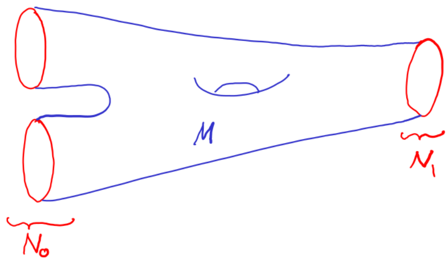
This is a $\mathbb{C}$-vector space. Since the path integral does not make mathematical sense in this generality anyway, we will not need to be careful with what kind

[1] There is by now a large zoo of mathematical objects which are all called "topological field theories". We will focus on non-extended TFTs (and so will largely avoid higher category theory), and oriented TFTs (and so avoid discussing various tangential structures).

[2] So $n-1$ space dimensions and one time dimension, though we will later only work in a purely euclidean setting with n space dimension and no time dimensions. In this sense we are rather modelling statistical field theory, not quantum field theory, but let us gloss over that.

of functions we mean here.[3] However, in Sections 3.3 and 4.5 we will see a finite and discrete variant of the path integral in the guise of statistical lattice models, which makes perfect sense.

Now suppose two $n-1$ manifolds N_0, N_1 are connected by an n-manifold M in the sense that the boundary of M is given by the union of N_0 and N_1:



Given a state $\Phi \in \mathcal{H}(N_0)$ we want to use M to determine its evolution to a state $\Phi' \in \mathcal{H}(N_1)$. This is done via the (in this generality ill-defined) path integral. Namely, to evaluate Φ' at a function $f_1 : N_1 \to T$, we formally set

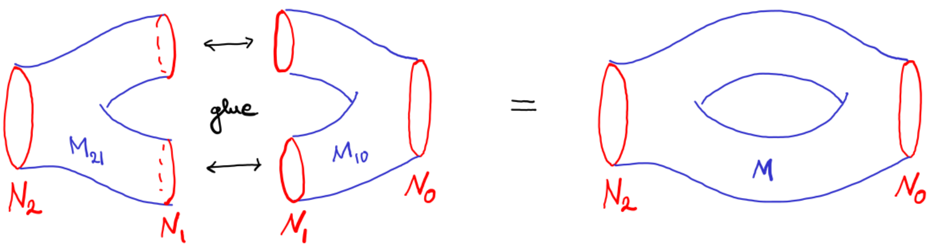
$$\Phi'(f_1) = \int_{f:M\to T, f|_{N_1}=f_1} Df \, \Phi(f|_{N_0}) \, e^{iS(f)} .$$

where the "integral" is over functions $f : M \to T$ which restrict to f_1 on the boundary component N_1 of M, with a "measure Df". The integrand contains of a weight $e^{iS(f)}$, where $S(f)$ is the action functional evaluated on f. (In the euclidean setting, this weight would be $e^{-E(f)}$ for E the energy.) Denote the resulting linear map by

$$Z(M) : \mathcal{H}(N_0) \to \mathcal{H}(N_1) ,$$

so that $\Phi' = Z(M)(\Phi)$.

Let us extract some properties we would expect Z and $\mathcal{H}$ to have. The main property is the following: Take $n-1$ manifolds N_0, N_1, N_2 and n-manifolds M_{10} with boundary N_0 and N_1, and M_{21} with boundary N_1 and N_2. Let M be the result of gluing M_{21} and M_{10} along N_1, so that M has boundaries N_0 and N_2:



[3]It is actually possible to mathematically define path integrals for certain euclidean two-dimensional quantum field theories – specifically Liouville conformal field theory – via probabilistic measures.

Then one formally would expect, for $f_2 : N_2 \to \mathbb{C}$,

$$\begin{aligned}
\Phi'(f_2) = \big(Z(M)(\Phi)\big)(f_2) &= \int_{f:M\to T, f|_{N_2}=f_2} Df\, \Phi(f|_{N_0})\, e^{iS(f)} \\
&= \int_{f_1:N_1\to T} \left(\int_{f:M\to T, f|_{N_2}=f_2, f|_{N_1}=f_1} Df\, \Phi(f|_{N_0})\, e^{iS(f)} \right) \\
&= \int_{f_1:N_1\to T} \left(\int_{g:M_{21}\to T, g|_{N_2}=f_2, g|_{N_1}=f_1} Dg\, e^{iS(g)} \int_{h:M_{10}\to T, h|_{N_1}=f_1} Dh\, \Phi(h|_{N_0})\, e^{iS(h)} \right) \\
&= \int_{f_1:N_1\to T} \left(\int_{g:M_{21}\to T, g|_{N_2}=f_2, g|_{N_1}=f_1} Dg\, e^{iS(g)} \Big(Z(M_{10})(\Phi)\Big)(f_1) \right) \\
&= \Big(Z(M_{21}) Z(M_{10})(\Phi)\Big)(f_2)
\end{aligned}$$

We can write this as

$$Z\big(M_{21} \circ_{\text{glue}} M_{10}\big) \;=\; Z(M_{21}) \circ_{\text{compose}} Z(M_{10}) \ .$$

This motivates why we want to use categories and functors to define topological field theories.

Next, let us look at the state spaces $\mathcal{H}_N$. Suppose N is a disjoint union $N = N_1 \sqcup N_2$ of two $n-1$ manifolds. Then giving a function on N is the same as giving a function f_1 on N_1 and a function f_2 on N_2, so that $\mathrm{Fun}(N, T)$ is the cartesian product

$$\mathrm{Fun}(N, T) = \mathrm{Fun}(N_1, T) \times \mathrm{Fun}(N_2, T) \ .$$

What does this mean for the state spaces? If N were a finite set, we would have

$$\mathrm{Fun}\big(\mathrm{Fun}(N, T), \mathbb{C}\big) \cong \mathrm{Fun}\big(\mathrm{Fun}(N_1, T), \mathbb{C}\big) \otimes_{\mathbb{C}} \mathrm{Fun}\big(\mathrm{Fun}(N_2, T), \mathbb{C}\big) \ .$$

(Why? What is the bilinear map $\mathrm{Fun}\big(\mathrm{Fun}(N_1, T), \mathbb{C}\big) \times \mathrm{Fun}\big(\mathrm{Fun}(N_2, T), \mathbb{C}\big) \to \mathrm{Fun}\big(\mathrm{Fun}(N, T), \mathbb{C}\big)$ underlying the map out of the tensor product?) ? For infinite N this may still hold for some variant of the tensor product, e.g. involving some completion, but ignoring such problems for this discussion, we would expect

$$\mathcal{H}(N_1 \sqcup N_2) \;=\; \mathcal{H}(N_1) \otimes_{\mathbb{C}} \mathcal{H}(N_2) \ .$$

This motivates why we use monoidal categories (categories with a product operation) and functors to define topological field theories.

One can come back to this motivation from time to time to see what the construction in the following would represent in the path integral setting. We will now leave the motivational part behind and turn to mathematics.

Bibliographic notes for Section 1

The formulation of topological field theory as a functor is due to Atiyah [At] and to Segal [Seg]. For the rigorous probabilistic formulation of the path integral in two-dimensional Liouville conformal field theory, see [GKRV]. There are many text books and lecture notes on topological field theory available. Some of the ones with a more mathematical focus and more closely related to the approach taken here are [BK, CR2, FSW, GP, Ko, Te, Tu, TV, Wa].

2 Categories, tensor products, braidings

In this section we collect the necessary ingredients from category theory to state the definition of a topological field theory.

2.1 Categories, functors, natural transformations

Definition 2.1. A *category* C consists of the following data:

- $\mathrm{ob}(C)$: a class of objects,
- for all objects $a, b \in \mathrm{ob}(C)$ (or just $a, b \in C$ for short) a set of *morphisms from a to b*,

$$\mathrm{Hom}_C(a, b) \qquad \text{(also write } C(a, b) \text{ for short)} \ ,$$

- for all $a \in C$ an *identity morphism* $1_a \in C(a, a)$,
- for all $a, b, c \in C$ a map called *composition*

$$\circ : C(b, c) \times C(a, b) \longrightarrow C(a, c) \quad , \quad (f, g) \longmapsto f \circ g \ .$$

These satisfy the condition (write $a \xrightarrow{f} b$ or $f : a \to b$ for $f \in C(a, b)$):

- (unit) for all $a \xrightarrow{f} b$: $1_b \circ f = f = f \circ 1_a$,
- (associativity) for all $a \xrightarrow{f} b$, $b \xrightarrow{g} c$, $c \xrightarrow{h} d$ we have

$$h \circ (g \circ f) = (h \circ g) \circ f \ \in \ C(a, d) \ .$$

■ Notation: We say two objects $a, b \in C$ are *isomorphic* if there are morphisms $f : a \to b$ and $g : b \to a$ such that $f \circ g = 1_b$ and $g \circ f = 1_a$. In this case we write $a \cong b$.

Example 2.2.
1. Set: objects are sets and morphisms are maps between sets.
2. Vec_k: vector spaces over a field k and k-linear maps.
3. R-mod: modules over a ring R and R-linear maps. If $R = k$ is a field, then k-mod is the same as Vec_k.
4. $\mathsf{Rep}(G)$: k-linear representations of a group G and intertwiners of representations, for a field k.

5. Vec_G: G-graded k-vector spaces and grade preserving linear maps:

   - objects are $V = \bigoplus_{g \in G} V_g$, where V_g is called the component of grade g of V.
   - morphisms $f : U \to V$ are linear maps such that $f(U_g) \subset V_g$.

   Note that at this point we did not use the group structure at all, and G may as well just be a set. The group structure will enter when we turn Vec_G into an example of a monoidal category below.

6. SVec_k: super-vector spaces over k and even linear maps, that is, $\mathbb{Z}_2$-graded vector spaces and grade preserving linear maps. The reason to give $\mathbb{Z}_2$-graded vector spaces a special name will be the braiding we will put on it below.

■ A category C is called *k-linear* (for a field k) if each Hom-space $C(a,b)$ is a k-vector space, and if the composition of morphisms is bilinear.
Of the examples above, Vec_k, $\mathsf{Rep}(G)$, Vec_G, and SVec_k are k-linear.

Definition 2.3. Let C be a category. The *opposite category C^{opp} of C* is the category with

- objects: $\text{ob}(C^{\text{opp}}) = \text{ob}(C)$,
- morphisms: for all $a, b \in C^{\text{opp}} = C$, $C^{\text{opp}}(a,b) = C(b,a)$,
- the same units as C,
- composition

$$\circ^{\text{opp}} : C^{\text{opp}}(b,c) \times C^{\text{opp}}(a,b) \to C^{\text{opp}}(a,c) \quad , \quad (f,g) \mapsto g \circ f \ ,$$

  where the composition on the right hand side is that of $g \in C(b,a)$ and $f \in C(c,b)$ in C. In other words

$$f \circ^{\text{opp}} g \ = \ g \circ f \ .$$

Definition 2.4. Let C, D be categories. The *cartesian product $C \times D$* is the category with

- objects: pairs (c,d) with $c \in \text{ob}(C), d \in \text{ob}(D)$; we will write $c \times d \in \text{ob}(C \times D) = \text{ob}(C) \times \text{ob}(D)$,
- morphisms: $(C \times D)(c \times d, c' \times d') = C(c,c') \times D(d,d')$; if $f : c \to c'$ and $g : d \to d'$ we write $f \times g : c \times d \to c' \times d'$,

?

- (What are units and composition?)

Definition 2.5. Let C, D be categories. A *covariant functor F from C to D* (or functor for short) consists of

- for all $a \in \mathrm{ob}(C)$ an object $F(a) \in \mathrm{ob}(D)$,
- for all $a, b \in \mathrm{ob}(C)$ a map of sets

$$C(a,b) \to D(Fa, Fb) \quad , \quad f \mapsto F(f) \ ,$$

such that

- for all $a \in \mathrm{ob}(C)$, $F(1_a) = 1_{F(a)} \in D(Fa, Fa)$,
- for all $a \xrightarrow{f} b$, $b \xrightarrow{g} c$ morphisms in C we have

$$F(g \circ f) \ = \ F(g) \circ F(f) \ \in \ D(Fa, Fc) \ .$$

A *contravariant functor F from C to D* has the same data and conditions as above, except that on morphisms $F(-) : C(a,b) \to D(Fb, Fa)$ and $F(g \circ f) = F(f) \circ F(g)$. Equivalently, it is a covariant functor $C^{\mathrm{opp}} \to D$ (or $C \to D^{\mathrm{opp}}$).

Example 2.6.

1. For any category C, one has the *identity functor* $\mathrm{Id} : C \to C$ which is the identity function on objects and morphisms.
2. Forgetful functors U which forget some of the underlying structure. For example $\mathsf{Vec}_k \to \mathsf{Set}$ forgets the action of k but remembers the underlying set; $\mathsf{Rep}(G) \to \mathsf{Vec}_k$ forgets the action of G but remembers the underlying vector space.
3. Hom-functors: For any category C and $a \in C$, $C(a, -) : C \to \mathsf{Set}$ is a covariant functor, and $C(-, a)$ a contravariant one.
   If C is k-linear, the Hom-functor takes values in Vec_k: $C(a, -) : C \to \mathsf{Vec}_k$, etc. (Details?) ?
4. Let $(-)^* : \mathsf{Vec}_k \to \mathsf{Vec}_k$ be given by taking the dual vector space and assigning the dual linear map. Then $(-)^*$ is a contravariant functor. Note that $(-)^* = \mathrm{Hom}_k(-, k)$.

■ A functor $F : C \to D$ between k-linear categories C, D is called *k-linear* if F is k-linear on Hom-spaces, that is, if for all $a, b \in C$, the map $C(a,b) \to D(Fa, Fb)$ is k-linear. (Which of the above functors are k-linear?) ?

■ Notation: A functor $F : C \to D$ is called

- *full* if it is surjective on Hom-sets, i.e. if for all $a, b \in C$ the map $F(-) : C(a, b) \to D(Fa, Fb)$ is surjective,

- *faithful* if it is injective on Hom-sets, i.e. if the above map is injective for all $a, b \in C$.

- *essentially surjective (on objects)* if for every $d \in D$ there is a $c \in C$ such that $F(c) \cong d$.

(Which of the above functors are full/faithful/essentially surjective?) ? (7.4.26)

Definition 2.7. Let C, D be categories and $F, G : C \to D$ be functors from C to D. A *natural transformation* φ from F to G is an assignment, for each $a \in C$,

$$\varphi_a : F(a) \to G(a) \qquad \text{(i.e. } \varphi_a \in D(Fa, Ga))$$

and such that, for all morphism $a \xrightarrow{f} b$ in C the following diagram – called *naturality square* – commmutes:

$$\begin{tikzcd} F(a) \arrow[r, "F(f)"] \arrow[d, "\varphi_a"'] & F(b) \arrow[d, "\varphi_b"] \\ G(a) \arrow[r, "G(f)"] & G(b) \end{tikzcd}$$

We say the family $(\varphi_a : Fa \to Ga)_{a \in C}$ is *natural in* a, and we use the following notations for natural transformations interchangeably:

$$\varphi : F \to G \quad , \quad \varphi : F \Rightarrow G \quad , \quad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\varphi"] & B \end{tikzcd}$$

Example 2.8. 1. Double duals: Let $(-)^{**} : \mathsf{Vec}_k \to \mathsf{Vec}_k$ be the covariant functor obtained by applying the dual functor twice. Then there is natural transformation $\delta_V : V \to V^{**}$. The linear map δ_V is always injective, and it is surjective iff V is finite-dimensional.

2. Write End(Id) for the natural endomorphisms of the identity functor on R-mod. Then End(Id) is itself a ring and is isomorphic, as a ring, to the centre $Z(R)$ of R.

3. Let A be an algebra over a field k, and let A-mod be the category of A-modules. It is k-linear. (Why)? Let $U : A\text{-}\mathsf{mod} \to \mathsf{Vec}_k$ be the forgetful functor. Then $\mathrm{End}(U) \cong A$ as a k-algebra. ?

Exercise 2.9. Provide the details for the above example. That is: Ⓔ

1. The map δ_V sends $v \in V$ to the linear functional $V^* \to k$, which takes $\varphi \in V^*$ to $\varphi(v)$. Show that it is a natural transformation. Show the injectivity/surjectivity claim in the example.

2. Proceed in steps to verify the claim in the second part of the example above:

   (a) Let f, g be natural endomorphisms of the identity functor of $R\text{-}\mathsf{mod}$, so that for each R-module M we get an R-linear map $f_M : M \to M$, etc. Show that $f_M + g_M$ and $f_M \circ g_M$ are also a natural endomorphisms of Id. (This turns End(Id) into an associative, unital ring.)

   (b) Let $z \in Z(R)$ and define $\varphi(z) \in \mathrm{End}(\mathrm{Id})$ via $\varphi(z)_M : M \to M$, $m \mapsto z.m$. Show that this is indeed a natural endomorphism. Show that φ is a ring homomorphism.

   (c) Provide an inverse to φ. *Hint:* Evaluate a given natural endomorphism η on the ring R, seen as a left module over itself, and set $z = \eta_R(1)$, where $1 \in R$ is the unit of R. Why is $z \in Z(R)$? For a general R-module M, and $m \in M$, consider the map $f : R \to M$, $r \mapsto r.m$. Why is this R-linear? Evaluate the naturality square for f.

3. Now show the third part of the example. *Hint:* This works along the same lines as the previous part of this exercise. Why is it now possible to pick a general algebra element when defining φ as in (2b), rather than an element of the centre?

Lemma 2.10. Let C, D be categories and $S \subset C$ a subcategory such that the embedding $S \hookrightarrow C$ is full and essentially surjective. Then:

1. For every functor $F : S \to D$ there is a functor $\tilde{F} : C \to D$ which extends F.

2. For two functors $G, H : C \to D$ denote by $G|_S$ and $H|_S$ the restriction to S. Every natural transformation $\phi : G|_S \Rightarrow H|_S$ extends uniquely to a natural transformation $\tilde{\phi} : G \Rightarrow H$.

Note that part 2 shows that the functor $\tilde{F}$ in part 1 is unique up to unique natural isomorphism. Indeed, if there is $G : C \to D$ such that $G|_S = F$, then the identity natural transformation extends uniquely to a natural isomorphism $G \Rightarrow \tilde{F}$.

Exercise 2.11. Prove the above lemma. *Hint:* Part 1: For each object $c \in C$ pick an object $s(c) \in S$ and an isomorphism $f_c : c \to s(c)$. Set $\tilde{F}(c) := F(s(c))$. What should $\tilde{F}$ do on morphisms? Part 2: Consider the naturality squares for the isomorphism $f_c : c \to s(c)$ to show existence and uniqueness.

Definition 2.12. A functor $F : C \to D$ is an *equivalence (of categories)* if there is a functor $G : D \to C$ together with natural isomorphism

$$\eta : \mathrm{Id}_D \Rightarrow F \circ G \quad , \quad \theta : G \circ F \Rightarrow \mathrm{Id}_C \ .$$

Proposition 2.13. A functor is an equivalence if and only if it is essentially surjective and fully faithful.

Proof. Let $F : C \to D$ be a functor.

$\Rightarrow$: *essentially surjective:* Let $d \in D$ be given. Let G and θ be as in Definition 2.12. Choose $c = G(d)$. Then $\eta_d^{-1} : F(G(d)) = F(c) \to d$ is an isomorphism.

faithful: follows from the naturality square for θ. (How?)

full: Let $h : F(a) \to F(b)$ be given. We need to find $f : a \to b$ so that $F(f) = h$. Take

$$f = \Big[a \xrightarrow{\theta_a^{-1}} GF(a) \xrightarrow{G(h)} GF(b) \xrightarrow{\theta_b} b \Big] \ .$$

Note that

$$1_{Fa} = \Big[F(a) \xrightarrow{\eta_{Fa}} FGF(a) \xrightarrow{F(\theta_a)} F(a) \xrightarrow{F(\theta_a^{-1})} FGF(a) \xrightarrow{\eta_{Fa}^{-1}} F(a) \Big] \ .$$

Now substitute these expressions into $F(f) \circ 1_{Fa}$ and use naturality and functoriality to show that this is equal to h. (Details? *Hint:* Start by writing the above expression for 1_a along the top of a commuting diagram and $F(f)$ along the right downwards vertical. Fill in the next columns using naturality of η^{-1} and θ^{-1}.)

$\Leftarrow$: Let $S \subset D$ be the full subcategory consisting of all objects of the form $F(c)$ for some $c \in C$. As F is essentially surjective, the embedding $S \to D$ is full and essentially surjective. For each $s \in S$ pick a $H(s) \in C$ such that $FH(s) = s$.

Let $f : s \to t$ be a morphism in S. As F is fully faithful, $F : C(Hs, Ht) \to S(FHs, FHt) = S(s,t)$ is an isomorphism. Define $H(f) = F^{-1}(f)$. This turns H into a functor $S \to C$. (Why is H compatible with units and composition?)

By construction, $FH = \mathrm{Id}_S$. Next we construct a natural isomorphism $\theta : HF \to \mathrm{Id}_C$. Let $c \in C$ be given and set $s = F(c) \in S$, $c' = HF(c) = H(s)$. Then also $F(c') = FHF(c) = F(c)$. Define $\theta_c : HF(c) = c' \to c$ as $\theta_c = F^{-1}(1_s)$, with $F : C(c', c) \to S(Fc', Fc) = S(s, s)$. (Why is this a natural isomorphism? *Hint:* The inverse to θ_c is given by the preimage of 1_s for $F : C(c, c') \to S(s, s)$. For naturality of θ, apply F to the naturality square and use that F is faithful.)

Finally, by Lemma 2.10, $H : S \to C$ extends to a functor $G : D \to C$. We have $GF = HF \cong \mathrm{Id}_C$ via θ. Conversely, $FG : D \to D$ is an extension of the embedding $S \hookrightarrow D$. But so is Id_D, and by Lemma 2.10, the two extensions are isomorphic. $\square$

2.2 Monoidal categories and duals

Definition 2.14. A *monoidal category* is a tuple $(C, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ where C is a category, and:

- *(tensor product)* $\otimes : C \times C \to C$ is a functor.
- *(tensor unit)* $\mathbb{1} \in C$ is an object.
- *(associator)* α is a natural isomorphism $\otimes \circ (\mathrm{Id} \times \otimes) \Rightarrow \otimes \circ (\otimes \times \mathrm{Id})$, both of which are functors $C \times C \times C \to C$. In other words, α consists of a family of isomorphisms
$$\alpha_{a,b,c} : a \otimes (b \otimes c) \to (a \otimes b) \otimes c \ ,$$
natural in $a, b, c \in C$.
- *(unitors)* $\lambda : \mathbb{1} \otimes (-) \Rightarrow \mathrm{Id}_C$ and $\rho : (-) \otimes \mathbb{1} \Rightarrow \mathrm{Id}_C$ are natural isomorphism. In other words, λ and ρ are a family of isomorphisms
$$\lambda_a : \mathbb{1} \otimes a \to a \quad , \quad \rho_a : a \otimes \mathbb{1} \to a \ ,$$
natural in $a \in C$.

This data is subject to the conditions:

- *(pentagon axiom)*
$$\begin{tikzcd}
a \otimes (b \otimes (c \otimes d)) \arrow[r, "1_a \otimes \alpha_{b,c,d}"] \arrow[d, "\alpha_{a,b,c\otimes d}"'] & a \otimes ((b \otimes c) \otimes d) \arrow[r, "\alpha_{a,b\otimes c,d}"] & (a \otimes (b \otimes c)) \otimes d \arrow[d, "\alpha_{a,b,c} \otimes 1_d"] \\
(a \otimes b) \otimes (c \otimes d) \arrow[rr, "\alpha_{a\otimes b,c,d}"'] & & ((a \otimes b) \otimes c) \otimes d
\end{tikzcd}$$

- *(triangle axiom)*

$$\begin{tikzcd} a \otimes (\mathbb{1} \otimes b) \arrow[rr, "\alpha_{a,\mathbb{1},b}"] \arrow[dr, "1_a \otimes \lambda_b"'] & & (a \otimes \mathbb{1}) \otimes b \arrow[dl, "\rho_a \otimes 1_b"] \\ & a \otimes b & \end{tikzcd}$$

We will abbreviate $C = (C, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$.

■ The α, λ, ρ in the definition of a monoidal category are called *coherence isomorphism.* A monoidal category is called *strict* if its coherence isomorphism are all identities. In this case $(a \otimes b) \otimes c = a \otimes (b \otimes c)$, and we can omit brackets around tensor products.

Exercise 2.15. 1. The reason that we only consider the triangle axiom with $\mathbb{1}$ in the middle is that then the other two hold automatically, that is, the triangles (E)

$$\begin{tikzcd} \mathbb{1} \otimes (a \otimes b) \arrow[rr, "\alpha_{\mathbb{1},a,b}"] \arrow[dr, "\lambda_{a\otimes b}"'] & & (\mathbb{1} \otimes a) \otimes b \arrow[dl, "\lambda_a \otimes 1_b"] \\ & a \otimes b & \end{tikzcd} \quad , \quad \begin{tikzcd} a \otimes (b \otimes \mathbb{1}) \arrow[rr, "\alpha_{a,b,\mathbb{1}}"] \arrow[dr, "1_a \otimes \rho_b"'] & & (a \otimes b) \otimes \mathbb{1} \arrow[dl, "\rho_{a\otimes b}"] \\ & a \otimes b & \end{tikzcd}$$

commute. Show one of these. *Hint:* For the left triangle, write out the pentagon axiom for $a, \mathbb{1}, c, d$ and apply the triangle axiom where possible. Then set $a = \mathbb{1}$.

2. $\lambda_{\mathbb{1}}$ and $\rho_{\mathbb{1}}$ are both isomorphism $\mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$. Show that they coincide. *Hint:* Combine the following identities: a) $1_{\mathbb{1}} \otimes \lambda_a = \lambda_{\mathbb{1}\otimes a} : \mathbb{1} \otimes (\mathbb{1} \otimes a) \to \mathbb{1} \otimes a$ (compose with the isomorphism λ_a and use naturality for the resulting identity); b) $(\lambda_{\mathbb{1}} \otimes 1_a) \circ \alpha_{\mathbb{1},\mathbb{1},a} = \lambda_{\mathbb{1}\otimes a}$ (part 1) c) $(\rho_{\mathbb{1}} \otimes 1_{\mathbb{1}}) \circ \alpha_{\mathbb{1},\mathbb{1},\mathbb{1}} = 1_{\mathbb{1}} \otimes \lambda_{\mathbb{1}}$ (Triangle axiom). Conclude first $\rho_{\mathbb{1}} \otimes 1_{\mathbb{1}} = \lambda_{\mathbb{1}} \otimes 1_{\mathbb{1}}$ and then $\rho_{\mathbb{1}} = \lambda_{\mathbb{1}}$.

■ A category is called *essentially small* if the isomorphism classes of objects form a set. For example, Vec_k is not essentially small (as each set can be the basis of a vector space), but the category $\mathsf{Vec}_k^{\mathrm{fd}}$ of *finite-dimensional* vector spaces is essentially small, with isomorphism classes of finite dimensional vector spaces being in bijection with natural numbers via the dimension.

Remark 2.16. A monoidal category is an example of a *categorification* of an algebraic structure: A *monoid* is a set M together with an associative product $\cdot : M \times M \to M$, and a unit $1 \in M$ for that product. Thus for

$a, b, c \in M$, we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ and $1 \cdot a = a = a \cdot 1$. In the (non-unique) process of categorification, such identities get replaced by new data – the coherence isomorphism – which in turn have to satisfy their own, new, identities. Conversely, *decategorification* amounts to passing to isomorphism classes of objects (in this example – other notions of decategorification exist). The decategorification of an essentially small monoidal category is a monoid. (Details?) ?

Example 2.17. (In the examples below, we mostly only specify the tensor product functor. What is the other data? In particular, what is the tensor unit in each case?) ?

1. Set: One can take $\otimes = \sqcup$ (disjoint union) or $\otimes = \times$ (cartesian product).

2. Vec_k: One can take $\otimes = \oplus$ (direct sum of vector spaces) or $\otimes = \otimes_k$ (tensor product of vector spaces).

3. R-mod: Suppose R is commutative and take $\otimes = \otimes_R$. (What fails if R is not commutative?) ?

4. $\mathsf{Rep}(G)$: For two representation R, S of G, we can take the tensor product $R \otimes_k S$ of the underlying vector spaces, with action, for $g \in G$, $g.(r \otimes_k s) := (g.r) \otimes_k (g.s)$.

5. Vec_G: For $U, V \in \mathsf{Vec}_G$ with graded components U_g, V_g, we set $U \otimes V$ to be the vector space $U \otimes_k V$ with graded components

$$(U \otimes V)_g = \bigoplus_{h \in G} U_{gh} \otimes_k V_{h^{-1}} \ .$$

   The tensor unit is $\mathbb{1} = k_e$, i.e. the one-dimensional vector space k, concentrated in degree $e \in G$.

6. SVec_k: still the same as $\mathbb{Z}_2$-graded vector spaces.

■ A monoidal category is *k-linear* if C is k-linear and if the functor $\otimes$ is k-linear in each entry. That is, for each $c \in C$, the functors $(-) \otimes c, c \otimes (-) : C \to C$ are k-linear. (Why is Vec_k with tensor product $\otimes_k$ a k-linear monoidal category, while with $\oplus$ it is not?) ?

Definition 2.18. Let C be a monoidal category and $a \in C$. A *left dual* of a is an object $a^* \in C$ together with morphisms

$$\mathrm{ev}_a : a^* \otimes a \to \mathbb{1} \quad , \quad \mathrm{coev}_a : \mathbb{1} \to a \otimes a^* \ ,$$

called *evaluation* and *coevaluation*, which satisfy the *zig-zag identities*, which states that the compositions

$$a \xrightarrow{\sim} \mathbb{1} \otimes a \xrightarrow{\mathrm{coev}_a \otimes 1_a} (a \otimes a^*) \otimes a \xrightarrow{\sim} a \otimes (a^* \otimes a) \xrightarrow{1_a \otimes \mathrm{ev}_a} a \otimes \mathbb{1} \xrightarrow{\sim} a \ , \ \text{and}$$
$$a^* \xrightarrow{\sim} a^* \otimes \mathbb{1} \xrightarrow{1_{a^*} \otimes \mathrm{coev}_a} a^* \otimes (a \otimes a^*) \xrightarrow{\sim} (a^* \otimes a) \otimes a^* \xrightarrow{\mathrm{ev}_a \otimes 1_{a^*}} \mathbb{1} \otimes a^* \xrightarrow{\sim} a^*$$

are equal to 1_a and 1_{a^*}, respectively. The arrows stand for the corresponding coherence isomorphism of C.
Analogously one defines a *right dual* of a to be an object ${}^*a$ together with morphisms

$$\widetilde{\mathrm{ev}}_a : a \otimes {}^*a \to \mathbb{1} \quad , \quad \widetilde{\mathrm{coev}}_a : \mathbb{1} \to {}^*a \otimes a \ ,$$

satisfying zig-zag identities. (Write out the two zig-zag identities in this case.) ?
A monoidal category in which each object has a left and a right dual is called *rigid*.

Example 2.19. 1. In Set with $\sqcup$, only the empty set has a left and a right dual (both being again the empty set). (Why? *Hint:* For a non-empty set X there is a unique map $\emptyset \to X$, but there is no map $X \to \emptyset$.) ?

2. In Vec_k, a vector space V has a left (resp. a right) dual if and only if it is finite-dimensional. In that case, both dual are given by the dual vector space $V^* = \mathrm{Hom}_k(V, k)$. (Why? What are the (co)evaluation maps?). In particular, the category of finite-dimensional vector space $\mathsf{Vec}_k^{\mathrm{fd}}$ is rigid. ?

The next lemma shows that having a left or right dual is a *property*, and not structure. Namely if duals exist, any two choices are canonically isomorphic.

Lemma 2.20. Let C be a monoidal category, $a \in C$, and let $(a^*, \mathrm{ev}_a, \mathrm{coev}_a)$ and $(a', \mathrm{ev}'_a, \mathrm{coev}'_a)$ be left duals for a. Then there is a unique isomorphism $f : a^* \to a'$ such that

$$\mathrm{ev}'_a \circ (f \otimes 1_a) = \mathrm{ev}_a \ .$$

This isomorphism automatically satisfies

$$(1_a \otimes f) \circ \mathrm{coev}_a = \mathrm{coev}'_a \ .$$

Equivalently, one can exchange the two equalities in the statement of the lemma. (8.4.26)

Exercise 2.21. Prove the above lemma. *Hint:* Use zig-zag identities to extract f from the above condition, write the only possible morphism that could be the inverse, and again use the zig-zag identity to see that it indeed is an inverse. E

Exercise 2.22. Let C be a monoidal category, $a, b, c \in C$, and suppose b has a left dual. Show that $C(a \otimes b, c) \cong C(a, c \otimes b^*)$. E

For info: The above exercise can be used in a more interesting example of dualisability: In R-mod the objects which have a left (or right) dual are precisely the finitely generated projective R-modules.
To see this, first suppose that M is finitely generated projective, or, equivalently, that M a direct summand of R^n for some n. Set $M^* = \mathrm{Hom}_R(M, R)$. The embedding and projection maps realising M have components $e_i : R \to M$ and $p_i : M \to R$, $i = 1, \dots, n$ such that $\sum_i e_i \circ p_i = \mathrm{id}_M$. The evaluation $M^* \otimes_R M \to R$ is just composition, and the coevaluation $R \to M \otimes_R M^*$ is $r \mapsto \sum_i e_i(r) \otimes_R p_i$. Note that for $R = k$ a field, this reproduces the corresponding maps for finite-dimensional k-vector spaces.
Now let M be an R-module which has a left dual M'. Then $\mathrm{Hom}_R(M, -) \cong \mathrm{Hom}_R(R, (-) \otimes_R M')$, and since R is projective and $\otimes_R$ is right exact, so is $\mathrm{Hom}_R(R, (-) \otimes_R M')$. Therefore $\mathrm{Hom}_R(M, -)$ is right exact, and so M is projective. The coevaluation map $\mathrm{coev}_M : R \to M \otimes_R M'$, evaluated on $1 \in R$, is of the form $\mathrm{coev}_M(1) = \sum_{j=1}^n m_j \otimes_R m'_j$ for some n, $m_j \in M$, $m'_j \in M'$. Hence the snake identity can only hold if the m_j generate M and the m'_j generate M', and so M must be finitely generated.

Definition 2.23. Let C, D be monoidal categories. A *monoidal functor* $C \to D$ is a tuple $F = (F, F^2, F^0)$ where

- $F : C \to D$ is a functor,
- F^2 is a natural isomorphism $F(-) \otimes^D F(-) \Rightarrow F(- \otimes^C -)$ of functors $C \times C \to D$. In other words, F^2 consists of a family of isomorphisms
$$F^2_{a,b} : F(a) \otimes^D F(b) \to F(a \otimes^C b) \ ,$$
natural in $a, b \in C$,
- F^0 is an isomorphism $\mathbb{1}^D \to F(\mathbb{1}^C)$,

such that the following diagram commutes for all $a, b, c \in C$,

$$\begin{tikzcd}
F(a) \otimes^D (F(b) \otimes^D F(c)) \arrow[r, "1 \otimes F^2_{b,c}"] \arrow[d, "\alpha^D_{F(a),F(b),F(c)}"'] & F(a) \otimes^D F(b \otimes^C c) \arrow[r, "F^2_{a,b\otimes c}"] & F(a \otimes^C (b \otimes^C c)) \arrow[d, "F(\alpha^C_{a,b,c})"] \\
(F(a) \otimes^D F(b)) \otimes^D F(c) \arrow[r, "F^2_{a,b} \otimes 1"'] & F(a \otimes^C b) \otimes^D F(c) \arrow[r, "F^2_{a\otimes b,c}"'] & F((a \otimes^C b) \otimes^C c)
\end{tikzcd}$$

and such for all $a \in C$:

$$\lambda^D_{F(a)} = \Big[\mathbb{1}^D \otimes^D F(a) \xrightarrow{F^0 \otimes 1_{F(a)}} F(\mathbb{1}^C) \otimes^D F(a) \xrightarrow{F^2_{\mathbb{1},a}} F(\mathbb{1}^C \otimes^C a) \xrightarrow{F(\lambda^C_a)} F(a)\Big]$$

$$\rho^D_{F(a)} = \Big[F(a) \otimes^D \mathbb{1}^D \xrightarrow{1_{F(a)} \otimes F^0} F(a) \otimes^D F(\mathbb{1}^C) \xrightarrow{F^2_{a,\mathbb{1}}} F(a \otimes^C \mathbb{1}^C) \xrightarrow{F(\rho^C_a)} F(a)\Big]$$

(What is the decategorification of a monoidal functor?) ?

Proposition 2.24. Let C, D be monoidal categories and $F : C \to D$ be a monoidal functor. If $c \in C$ has a left (resp. right) dual in C, then $F(c)$ has a left (resp. right) dual in D.

Exercise 2.25. Show the above proposition for either left duals or right duals. *Hint:* If c^* is a left dual of c in C, the evaluation needed to establish $F(c^*)$ as a left dual of $F(c)$ can be obtained as E

$$F(c^*) \otimes F(c) \to F(c^* \otimes c) \to F(\mathbb{1}) \to \mathbb{1} \ .$$

Definition 2.26. Let C, D be monoidal categories, and $F, G : C \to D$ monoidal functors. A *monoidal natural transformation* is a natural transformation $\eta : F \Rightarrow G$ such that the following diagrams commute for all $a, b \in C$:

$$\begin{tikzcd} F(a) \otimes^D F(b) \arrow[r, "F^2_{a,b}"] \arrow[d, "\eta_a \otimes \eta_b"'] & F(a \otimes^C b) \arrow[d, "\eta_{a \otimes b}"] \\ G(a) \otimes^D G(b) \arrow[r, "G^2_{a,b}"'] & G(a \otimes^C b) \end{tikzcd} \quad , \quad \begin{tikzcd} \mathbb{1}^D \arrow[r, "F^0"] \arrow[dr, "G^0"'] & F(\mathbb{1}^C) \arrow[d, "\eta_{\mathbb{1}}"] \\ & G(\mathbb{1}^C) \end{tikzcd}$$

■ Let C be a monoidal category. We can use C to build a new monoidal category C^{str}, the *strictification of* C, which is strict monoidal. C^{str} has:

- objects: finite lists $\underline{a} = (a_1, a_2, \dots, a_n)$ of objects $a_i \in C$. The empty list $()$ is allowed. For each $\underline{a} \in C^{\text{str}}$ we can define an object $T(\underline{a}) \in C$ as
$$T(\underline{a}) = a_1 \otimes (a_2 \otimes (a_3 \otimes \cdots \otimes a_n)) \cdots) \ ,$$
and we declare $T(\,()\,) = \mathbb{1}$.
- morphisms: We define $C^{\text{str}}(\underline{a}, \underline{b}) = C(T(a), T(b))$. Composition and units are inherited from C in this way.

Note that by construction $T : C^{\text{str}} \to C$ is a fully faithful and essentially surjective functor, and hence gives an equivalence of categories by Proposition 2.13. Next we turn C^{str} into a monoidal category:

- The tensor product $*$ is defined on objects as the concatenation of lists,

$$\underline{a} * \underline{b} = (a_1, \dots, a_m, b_1, \dots, b_n) \ .$$

- To define the tensor product of morphisms, we first define an isomorphism

$$T^2_{\underline{a},\underline{b}} : T(\underline{a}) \otimes T(\underline{b}) \to T(\underline{a} * \underline{b}) \ .$$

in C. If one of the lists $\underline{a}$, $\underline{b}$ has length zero, we set

$$T^2_{(),\underline{b}} = \Big[T(()) \otimes T(\underline{b}) = \mathbb{1} \otimes T(\underline{b}) \xrightarrow{\lambda_{T(\underline{b})}} T(\underline{b}) \Big] \ ,$$
$$T^2_{\underline{a},()} = \Big[T(\underline{a}) \otimes T(()) = T(\underline{a}) \otimes \mathbb{1} \xrightarrow{\rho_{T(\underline{a})}} T(\underline{a}) \Big] \ .$$

This is unambiguous for $\underline{a} = () = \underline{b}$ by Exercise 2.15 (2). If both lists $\underline{a}$, $\underline{b}$ have positive length, $T^2_{\underline{a},\underline{b}}$ is the isomorphism

$$\begin{aligned} &(a_1 \otimes (a_2 \otimes \cdots \otimes a_m) \cdots)) \otimes (b_1 \otimes (b_2 \otimes \cdots \otimes b_m) \cdots)) \\ &\to a_1 \otimes (a_2 \otimes \cdots \otimes (a_m \otimes (b_1 \otimes (b_2 \otimes \cdots \otimes b_m) \cdots)) \ , \end{aligned}$$

obtained by repeated application of associators. The first step is

$$\begin{aligned} &\big\{ a_1 \otimes (a_2 \otimes \cdots \otimes a_m) \cdots) \big\} \otimes T(\underline{b}) \\ &\xrightarrow{\alpha^{-1}} a_1 \otimes \big\{ (a_2 \otimes \cdots \otimes a_m) \cdots)) \otimes T(\underline{b}) \big\} \ . \end{aligned}$$

(What are the three objects indexing the associator α^{-1} above? Draw a bracketing as a tree and visualise the above rebracketing in that way. Given an inductive definition of T^2.) ?

- The tensor product of morphisms $f : \underline{a} \to \underline{a}'$ and $g : \underline{b} \to \underline{b}'$ is

$$f * g = \Big[T(\underline{a} * \underline{b}) \xrightarrow{(T^2_{\underline{a},\underline{b}})^{-1}} T(\underline{a}) \otimes T(\underline{b}) \xrightarrow{f \otimes g} T(\underline{a}') \otimes T(\underline{b}') \xrightarrow{T^2_{\underline{a}',\underline{b}'}} T(\underline{a}' * \underline{b}') \Big] \ .$$

- The tensor unit is the empty list, $\mathbb{1} = ()$.

- The associator and unitors of C^{str} are identity natural transformations.

Theorem 2.27. There is a monoidal equivalence $C^{\mathrm{str}} \to C$ whose underlying functor is T.

Proof (sketch). We already saw that T gives an equivalence of categories. We now claim that T^2 as above and $T^0 = 1_{\mathbb{1}}$ equip T with the structure of a monoidal functor.

We need to check the two coherence diagrams in Definition 2.23 for T. Compatibility with the unitors holds by construction. For example,

$$\lambda_{T(\underline{a})} = \Big[\mathbb{1} \otimes T(\underline{a}) \xrightarrow{T^0 \otimes 1_{T(\underline{a})}} T(()) \otimes T(\underline{a}) \xrightarrow{T^2_{(),\underline{a}}} T(() * \underline{a}) \xrightarrow{T(1_{\underline{a}})} T(\underline{a})\Big]$$

is just the definition of $T^2_{(),a}$. Compatibility with the associator amounts to the identity, for all $\underline{a}$, $\underline{b}$, $\underline{c}$,

$$T^2_{\underline{a},\underline{b}*\underline{c}} \circ (1_{\underline{a}} \otimes T^2_{\underline{b},\underline{c}}) = T^2_{\underline{a}*\underline{b},\underline{c}} \circ (T^2_{\underline{a},\underline{b}} \otimes 1_{\underline{c}}) \circ \alpha_{T(\underline{a}),T(\underline{b}),T(\underline{c})} \ .$$

This follows from the coherence conditions of C by induction. We skip the details and refer to [Ka, Lem. XI.5.2]. □

■ The above theorem implies MacLane's Coherence Theorem for monoidal categories which states that any two ways to achieve a rebracketing or adding/ removing tensor units in a product of objects via the coherence isomorphisms of C result in the same isomorphism. (Why?) ?

■ It is often useful to carry out proofs of identities in a strict monoidal category as expressions with coherence isomorphisms get quite long. One then concludes that the corresponding identity with coherence isomorphisms added holds as well by mapping it to an equivalent strict category. (9.4.26)

Lemma 2.28. Let C be a monoidal category. Then $\mathrm{End}(\mathbb{1})$ is a commutative monoid, that is, for $f, g : \mathbb{1} \to \mathbb{1}$, we have $f \circ g = g \circ f$. If C is in addition k-linear, then $\mathrm{End}(\mathbb{1})$ is a commutative k-algebra.

Proof. It only remains to show that $\mathrm{End}(\mathbb{1})$ is commutative. We will show this from the identity $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$, and naturality of λ and ρ,

$$\begin{aligned} f \circ g \circ \lambda_{\mathbb{1}} &= f \circ \lambda_{\mathbb{1}} \circ (1_{\mathbb{1}} \otimes g) = f \circ \rho_{\mathbb{1}} \circ (1_{\mathbb{1}} \otimes g) = \rho_{\mathbb{1}} \circ (f \otimes g) \\ &= \lambda_{\mathbb{1}} \circ (f \otimes g) = g \circ \lambda_{\mathbb{1}} \circ (f \otimes 1_{\mathbb{1}}) = g \circ \rho_{\mathbb{1}} \circ (f \otimes 1_{\mathbb{1}}) \\ &= g \circ f \circ \rho_{\mathbb{1}} = g \circ f \circ \lambda_{\mathbb{1}} \end{aligned}$$

Since $\lambda_{\mathbb{1}}$ is an isomorphism, we have $f \circ g = g \circ f$. □

2.3 Braidings

■ For categories C, D, write $\mathrm{swap} : C \times D \to D \times C$ for the functor that swaps objects and morphisms between the left and right argument: for $c \in C, d \in D$ we set $\mathrm{swap}(c, d) = (d, c)$ and analogously for morphisms.

Definition 2.29. Let C be a monoidal category. A *braiding* on C is a natural isomorphism

$$\beta \,:\, \otimes \Longrightarrow \otimes \circ \mathrm{swap} \ ,$$

that is, a family of isomorphism

$$\beta_{a,b} \,:\, a \otimes b \longrightarrow b \otimes a \ ,$$

natural in a and b, such that β satisfies the two *hexagon identities*

$$\begin{tikzcd} & (a \otimes b) \otimes c \arrow[r, "\beta_{a\otimes b,\, c}"] & c \otimes (a \otimes b) \arrow[dr, "\alpha_{c,a,b}"] & \\ a \otimes (b \otimes c) \arrow[ur, "\alpha_{a,b,c}"] \arrow[dr, "1_a \otimes \beta_{b,c}"'] & & & (c \otimes a) \otimes b \\ & a \otimes (c \otimes b) \arrow[r, "\alpha_{a,c,b}"'] & (a \otimes c) \otimes b \arrow[ur, "\beta_{a,c} \otimes 1_b"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} & a \otimes (b \otimes c) \arrow[r, "\beta_{a,\, b\otimes c}"] & (b \otimes c) \otimes a \arrow[dr, "\alpha^{-1}_{b,c,a}"] & \\ (a \otimes b) \otimes c \arrow[ur, "\alpha^{-1}_{a,b,c}"] \arrow[dr, "\beta_{a,b} \otimes 1_c"'] & & & b \otimes (c \otimes a) \\ & (b \otimes a) \otimes c \arrow[r, "\alpha^{-1}_{b,a,c}"'] & b \otimes (a \otimes c) \arrow[ur, "1_b \otimes \beta_{a,c}"'] & \end{tikzcd}$$

A braiding β is called *symmetric* if it squares to the identity in the sense that for all $a, b \in C$,

$$\beta_{b,a} \circ \beta_{a,b} = 1_{a\otimes b}$$

A *braided monoidal category* is a monoidal category equipped with a braiding. A *symmetric monoidal category* is a braided monoidal category whose braiding is symmetric.

Exercise 2.30. It follows from the axioms of a braided monoidal category that the braiding of any object with the tensor unit is determined by unitors: E

$$\beta_{\mathbb{1},a} = \Big[\mathbb{1} \otimes a \xrightarrow{\lambda_a} a \xrightarrow{\rho_a^{-1}} a \otimes \mathbb{1} \Big] \quad , \quad \beta_{a,\mathbb{1}} = \Big[a \otimes \mathbb{1} \xrightarrow{\rho_a} a \xrightarrow{\lambda_a^{-1}} \mathbb{1} \otimes a \Big] \ .$$

Show one of these identities. *Hint:* In the hexagon identities, set $b = \mathbb{1}$, and fill in the diagram with commuting squares and triangles by applying unitors $a \otimes (\mathbb{1} \otimes c) \xrightarrow{\sim} a \otimes c$, etc.

Example 2.31. • $\mathsf{Set}, \sqcup$ with exchange of arguments of the disjoint union: $\beta_{X,Y} : X \sqcup Y \to Y \sqcup X$. This braiding is symmetric. (How would you write this in terms of elements? Can you think of something less confusing than $\beta_{X,Y}(x) = x$?) ?

- $\mathsf{Vec}_k, \otimes_k$ with flip τ of tensor factors, $\beta_{U,V} = \tau_{U,V} : U \otimes_k V \to V \otimes_k U$, $\tau(u \otimes v) = v \otimes u$. This braiding is symmetric: $\tau_{V,U} \circ \tau_{U,V} = \mathrm{id}_{U \otimes V}$. Because β is natural, and because every object is a direct sum of copies of the tensor unit, the braiding on Vec_k is unique. (Details? *Hint:* For vector spaces U, V and elements $u \in U$, $v \in V$, consider the maps $f : k \to U$, $x \mapsto xu$, and $g : k \to V$, $x \mapsto xv$. Use naturality to conclude $\beta_{U,V}(u \otimes_k v) = v \otimes_k u$.) ?

- Rep_G: The flip $\tau_{U,V}$ of tensor factors of the underlying vector spaces provides a braiding.

- Vec_G: This category can be equipped with a braiding if and only if G is abelian (Why?). If G is abelian, a possible choice of braiding is the tensor flip $\tau_{U,V}$ of the underlying vector spaces. We will see in the next section that more choices are possible. ?

- SVec_k: Finally we can see the difference to $\mathsf{Vec}_{\mathbb{Z}_2}$. For $U = U_0 \oplus U_1$ and $V = V_0 \oplus V_1$ super-vector spaces, let $u \in U$, $v \in V$ be homogeneous elements. This means that $u \in U_0$ or $u \in U_1$, etc. We write $|u| \in \mathbb{Z}_2$ for the degree, or *parity*, of a homogeneous element. Then
$$\beta_{U,V} : U \otimes_k V \to V \otimes_k U \quad , \quad u \otimes v \mapsto (-1)^{|u|\,|v|}\, v \otimes u \ .$$
Thus, if both u and v have degree 1 (are "parity-odd"), then braiding them past each other produces a minus sign ("parity sign"). Note that this braiding is still symmetric, $\beta_{V,U} \circ \beta_{U,V} = \mathrm{id}_{U \otimes V}$, as the parity signs cancel.

In these examples, all braidings are symmetric. In the next section, we meet some non-symmetric braidings.

Definition 2.32. Let C, D be braided monoidal category. A *braided monoidal functor* from C to D is a monoidal functor $F : C \to D$ which satisfies, for all $a, b \in C$:

$$\begin{tikzcd}
F(a) \otimes^D F(b) \arrow[r, "\beta^D_{Fa,Fb}"] \arrow[d, "F^2_{a,b}"] & F(b) \otimes^D F(a) \arrow[d, "F^2_{b,a}"] \\
F(a \otimes^C b) \arrow[r, "F(\beta^C_{a,b})"] & F(b \otimes^C a)
\end{tikzcd}$$

■ A braided monoidal natural transformation between two braided monoidal functors is simply a monoidal natural transformation. That is, there is no additional compatibility condition for a monoidal natural transformation to preserve the braiding. (Write down all the plausible diagrams you can think ? of and check that they hold automatically.)

■ Let C be a braided monoidal category. Let C^{str} be its strictification as a monoidal category as in Theorem 2.27. We use the diagram in the above definition to define a braiding on C^{str}. Recall that

$$C^{\mathrm{str}}(\underline{a} * \underline{b}, \underline{b} * \underline{a}) \;=\; C(T(\underline{a} * \underline{b}), T(\underline{b} * \underline{a})) \ .$$

We define

$$\beta^{\mathrm{str}}_{\underline{a},\underline{b}} = \Big[T(\underline{a} * \underline{b}) \xrightarrow{(T^2_{\underline{a},\underline{b}})^{-1}} T(\underline{a}) \otimes T(\underline{b}) \xrightarrow{\beta_{T(\underline{a}),T(\underline{b})}} T(\underline{b}) \otimes T(\underline{a}) \xrightarrow{T^2_{\underline{b},\underline{a}}} T(\underline{b} * \underline{a}) \Big] \ .$$

In this way, the monoidal functor $T : C^{\mathrm{str}} \to C$ is by construction braided. The hexagon identities for β^{str} hold by construction, too. Indeed, a braided monoidal functor turns the hexagon identity of the source category into the hexagon identity of the target category. (Check this.) If the functor is ? faithful, the source hexagons hold if and only if the target hexagons hold.

In a strict category, the hexagons simplify as we can omit the associators. For example the first hexagon becomes:

$$\begin{tikzcd}
\underline{a} * \underline{b} * \underline{c} \arrow[rr, "\beta^{\mathrm{str}}_{\underline{a}*\underline{b},\, \underline{c}}"] \arrow[dr, "1_{\underline{a}} \otimes \beta^{\mathrm{str}}_{\underline{b},\underline{c}}"'] & & \underline{c} * \underline{a} * \underline{b} \\
& \underline{a} * \underline{c} * \underline{b} \arrow[ur, "\beta^{\mathrm{str}}_{\underline{a},\underline{c}} \otimes 1_{\underline{b}}"'] &
\end{tikzcd}$$

■ Let B be a braided monoidal category. If $a \in B$ has a left dual a^*, then a^* is also a right dual, and vice versa. (Details? *Hint:* If $\mathrm{ev}_a : a^* \otimes a \to \mathbb{1}$? is the evaluation map, take $\widetilde{\mathrm{ev}}_a = \mathrm{ev}_a \circ \beta_{a,a^*} : a \otimes a^* \to \mathbb{1}$. What does one have to choose for $\widetilde{\mathrm{coev}}_a$? Why do the zig-zag identities hold (pass to the strictification to avoid long diagrams).)

■ Let S be a symmetric monoidal category, and let $a \in S$ be an object with a left dual a^*. Define the *dimension* $\dim(a) \in \mathrm{End}(\mathbb{1})$ as follows:

$$\dim(a) = \Big[\mathbb{1} \xrightarrow{\mathrm{coev}_a} a \otimes a^* \xrightarrow{\beta_{a,a^*}} a^* \otimes a \xrightarrow{\mathrm{ev}_a} \mathbb{1} \Big] \ .$$

One can analogously define the dimension in terms of a right dual ${}^*a$. The next proposition shows that this does not matter, and that $\dim(-)$ is compatible with the tensor product.

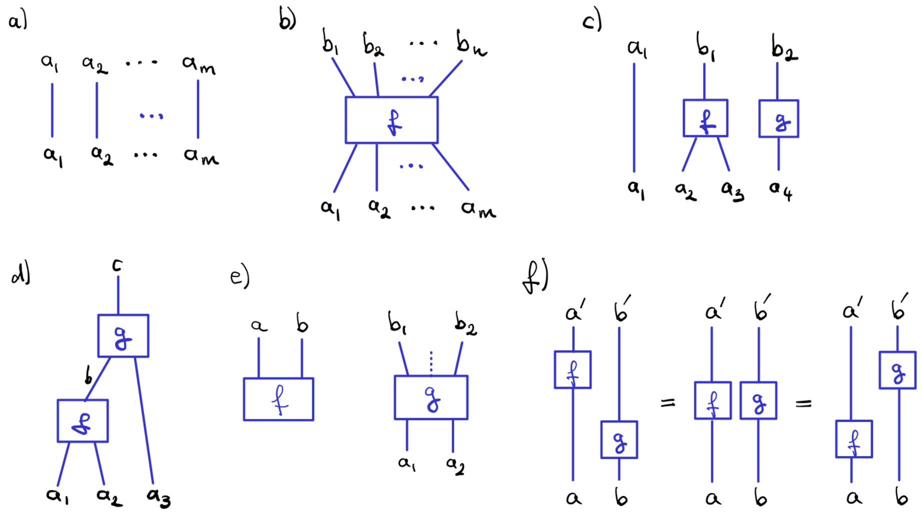


Figure 2.1: Graphical notation for monoidal categories

Proposition 2.33. $\dim(a)$ is independent of which left or right dual object of a one uses in its definition. It satisfies

$$\dim(a \otimes b) = \dim(a) \circ \dim(b) \ .$$

In order to prove this, we could draw a (very) big commuting diagram, or write a moderately long but hard to read chain of equalities in the strictification. Instead, we take this as an excuse to introduce the graphical calculus for braided monoidal categories, and give the proof of this proposition at the end of the next section as an illustration.

■ The multiplicativity of the above proposition does in general *not* hold if S is braided but not symmetric. We will see how to modify the definition of right duals in terms of left duals in order to fix this when we discuss ribbon structures (Section 5.1).

2.4 Graphical calculus for braided monoidal categories

Let C be a strict monoidal category. Write a tensor product $a_1 \otimes ... \otimes a_m$ by just writing $a_1 \cdots a_m$ next to each other.

- Write the identity morphism on a tensor product as vertical lines (Figure 2.1 a).
- A morphism $f : a_1 \otimes \cdots \otimes a_m \to b_1 \otimes \cdots \otimes b_n$ is a box, called *coupon*, labelled f with m vertical legs attached from below, labelled $a_1, \dots, a_m$ and n legs above, labelled $b_1, \dots, b_n$, see Figure 2.1 b

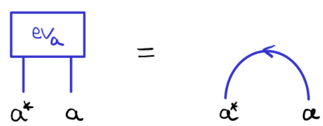
- For the tensor product of morphisms, write the two diagrams next to each other. For example, for $f : a_2 \otimes a_3 \to b_1$ and $g : a_4 \to b_2$, the product
$$a_1 \otimes a_2 \otimes a_3 \otimes a_4 \xrightarrow{1_{a_1} \otimes f \otimes g} a_1 \otimes b_1 \otimes b_2$$
is shown in Figure 2.1 c.

- Composition of morphisms is vertical stacking. For example, for $f : a_1 \otimes a_2 \to b$ and $g : b \otimes a_3 \to c$, the composition
$$a_1 \otimes a_2 \otimes a_3 \xrightarrow{f \otimes 1_{a_3}} b \otimes a_3 \xrightarrow{g} c$$
is shown in Figure 2.1 d.

- The tensor unit is not drawn. Or, if that is too confusing, it can be shown as a dashed line, which is allowed to end without being attached to a coupon (representing the unitor, which is the identity in a strict category). As an example, $f : \mathbb{1} \to a \otimes b$ and $g : a_1 \otimes a_2 \to b_1 \otimes \mathbb{1} \otimes b_2 = b_1 \otimes b_2$ are shown in Figure 2.1 e.

The strength of the graphical calculus is that we can deform the diagram in certain ways without changing the morphism it represents. For example, functoriality of the tensor product implies the equalities, for $f : a \to a'$ and $g : b \to b'$,
$$(f \otimes 1_{b'}) \circ (1_a \otimes g) \;=\; f \otimes g \;=\; (1_{a'} \otimes g) \circ (f \otimes 1_b) \;.$$
The representation in graphical calculus is given in Figure 2.1 f. Each valid diagram in graphical calculus can be turned into a sequence of morphisms by slicing the diagram horizontally in such a way that no coupons are cut.

■ Suppose $a \in C$ has a left dual a^* and/or a right dual ${}^*a$. The graphical presentation of the corresponding (co)evaluation maps are shown in Figure 2.2 a,c. If it is clear which (co)evaluation maps are used, the labels can be omitted. Note that these pictures are just shorthands for coupons, e.g.



The zig-zag identities for left and right duals are shown in Figures 2.2 b,d.

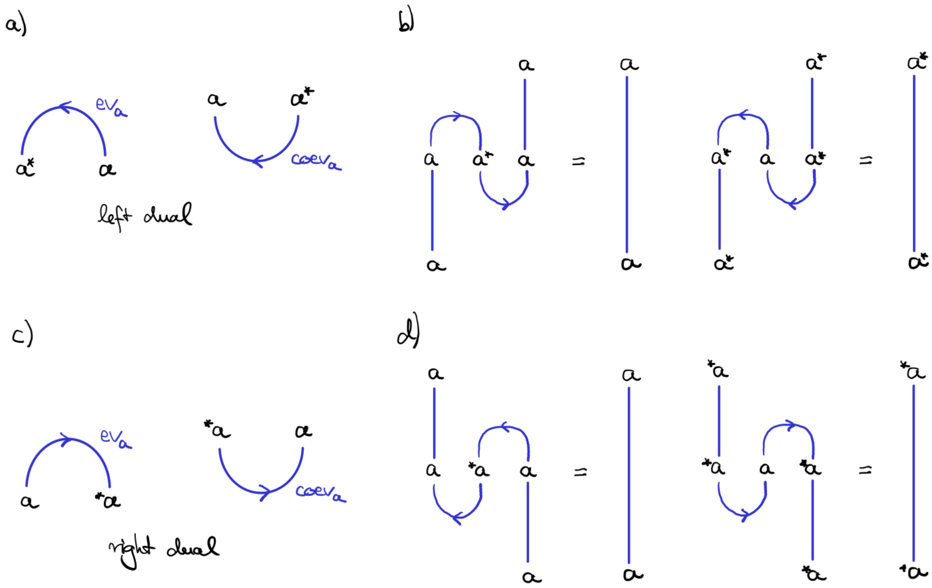


Figure 2.2: Graphical notation for left and right duals.

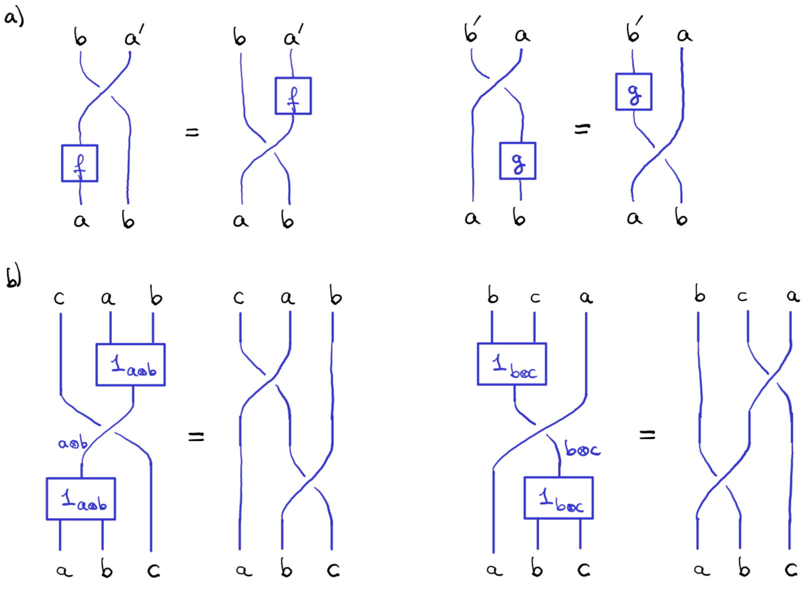
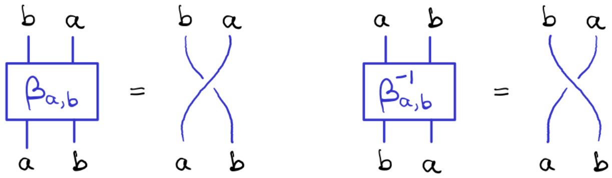
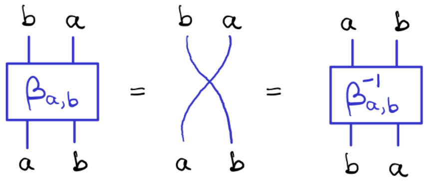


Figure 2.3: a) Naturality of the braiding. b) The two hexagon identities.

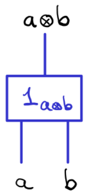
■ Suppose now that C is in addition braided. The pictures for the braiding and the inverse braiding are



If C is symmetric, the braiding and its inverse coincide, and we just draw a crossing:



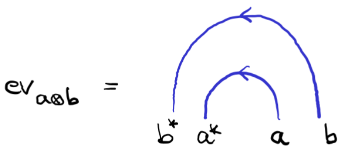
Naturality of the braiding is shown in Figure 2.3 a. The hexagon identities are so much part of the graphical calculus that they are hard to draw. But we can if we use an identity-coupon $a \otimes b \to a \otimes b$ where we write the incoming $a \otimes b$ as two separate strands, and the outgoing one has a single strand:

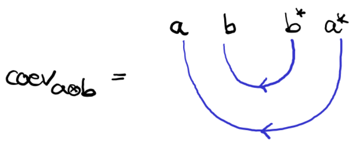


Using this, the two hexagons are shown in Figure 2.3 b.

■ We can now return to the proof of the proposition about multiplicativity of the dimension.

Proof of Proposition 2.33. Independence is an exercise. For multiplicativity with respect to the tensor product, we pass to the strictification S^{str} of S and carry out the computation in graphical calculus. First, we choose $b^* \otimes a^*$ as left dual of $a \otimes b$, with (co)evaluation maps





Then the proposition follows from the computation in Figure 2.4. The abbreviation "YBE" stands for "Yang-Baxter equation" and is given in Exercise 2.35. □

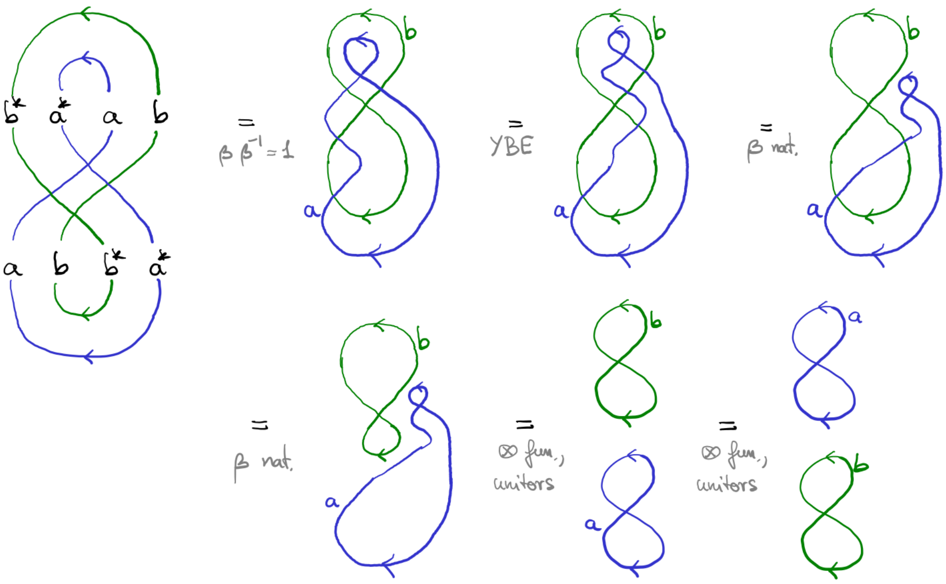


Figure 2.4: Graphical computation showing $\dim(a \otimes b) = \dim(a) \dim(b)$.

Exercise 2.34. Ⓔ

1. Show independence of the dimension of defining it in terms of left or right duals, and of the choice of dual object.

2. Compute the dimension of a finite dimensional vector space and of a super-vector space.

Exercise 2.35. Let C be a strict monoidal braided category. Ⓔ

1. The *Yang-Baxter equation* is the following identity between two morphisms $a \otimes b \otimes c \to c \otimes b \otimes a$:

   $$(\beta_{b,c} \otimes 1_a) \circ (1_b \otimes \beta_{a,c}) \circ (\beta_{a,b} \otimes 1_c) = (1_c \otimes \beta_{a,b}) \circ (\beta_{a,c} \otimes 1_b) \circ (1_a \otimes \beta_{b,c})$$

   Show this. (Should be short.)

2. The *braid group* B_n on n strands is the group with generators $\sigma_1, \dots, \sigma_{n-1}$ and relations

   $$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| \geq 2, \text{ and}$$
   $$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i = 1, \dots, n-2.$$

   Show that taking σ_i to be the braiding, for each $a \in C$ one obtains a group homomorphism $B_n \to \mathrm{End}(a^{\otimes n})$.

3. The *symmetric group* S_n in n elements is just the permutation group, i.e. the bijections $\{1,\dots,n\} \to \{1,\dots,n\}$. A presentation of the symmetric group is obtained by the same relation as for the braid group, together with the additional relation that $\sigma_i^2 = 1$ for $i = 1,\dots,n-1$.

   Suppose that the braiding of C is symmetric. Show that for each $a \in C$ one obtains a group homomorphism $S_n \to \mathrm{End}(a^{\otimes n})$.

2.5 G-graded vector spaces and group cohomology

In order to have some examples of non-trivial associators and non-symmetric braidings at hand, we make a little excursion into group cohomology.

Let G be a group, and let A be an abelian group equipped with a left action of G, that is, we have a group homomorphism $G \to \mathrm{Aut}(A)$. We write the action of $g \in G$ on $a \in A$ as $g.a$.

■ The *n-cochains* with values in A are simply

$$C^n(G,A) = \mathrm{Fun}(G^{\times n}, A) \ .$$

For $n = 0$ it is understood that $G^{\times 0} = \{*\}$, a singleton set, so that $C^0(G,A) = A$. In words, a degree-n cochain is a function from n copies of G to A, without any conditions.

■ Next we define the *differential* or *coboundary map* $d^n : C^n(G,A) \to C^{n+1}(G,A)$ as

$$\begin{aligned}(d\varphi)(g_1,g_2,\dots,g_{n+1}) = g_1.\varphi(g_2,\cdots,g_{n+1}) + \sum_{k=1}^{n}(-1)^k\varphi(\cdots,g_kg_{k+1},\cdots) \\ + (-1)^{n+1}\varphi(g_1,\dots,g_n)\end{aligned}$$

For the first few values of n, this reads

$$\begin{aligned}(d\varphi^0)(g_1) &= g_1.\varphi^0 - \varphi^0 \ , \\ (d\varphi^1)(g_1,g_2) &= g_1.\varphi^1(g_2) - \varphi^1(g_1g_2) + \varphi^1(g_1) \ , \\ (d\varphi^2)(g_1,g_2,g_3) &= g_1.\varphi^2(g_2,g_3) - \varphi^2(g_1g_2,g_3) + \varphi^2(g_1,g_2g_3) - \varphi^2(g_1,g_2) \ .\end{aligned}$$

(Check that indeed $d^2 = 0$.) ?

Definition 2.36. Let G be a group acting on an abelian group A from the left. The *n-cocycles*, and *n'th cohomology* in group cohomology are (14.4.26)

$$Z^n(G,A) = \ker d^n \quad , \quad H^n(G,A) = \ker d^n / \mathrm{im}\, d^{n-1} \ .$$

with differential as given above. An n-cocycle $\varphi \in Z^n(G,A)$ is called *normalised* if

$$\varphi(g_1,\dots,g_n) = 0 \text{ whenever at least one } g_i = 1.$$

For example, 0-cocycles are the same as the 0'th cohomology and are given by

$$H^0(G,A) = Z^0(G,A) = A^G = \{a \in A | g.a = a \text{ for all } g \in G\} \ ,$$

the G-invariant subgroup of A.

We will import the following result from homological algebra without proof (see [HS, Sec. VI.13]).

Lemma 2.37. For every n-cocycle $\varphi \in Z^n(G,A)$ there is a normalised n-cocycle ψ such that ψ and φ differ by a coboundary. Equivalently, every cohomology class has a normalised representative.

For info: For cyclic groups there is a simple expression to compute group cohomology in every degree [Wei, Thm. 6.2.2]: Let C_m be the cyclic group with m elements, with generator σ and group operation written multiplicatively, $\sigma^m = e$. We define group homomorphism $E, N : A \to A$ as, for $a \in A$,

$$E(a) = \sigma.a - a \quad , \quad N(a) = a + \sigma.a + \sigma^2.a + \dots + \sigma^{m-1}.a \ .$$

Note that $N(E(a)) = 0$ and that $N(a) \in A^G$. The group cohomology of C_m with values in A is given by

$$H^n(C_m, A) = \begin{cases} A^G & ; n = 0 \\ \ker N / \operatorname{im} E & ; n = 1, 3, 5, \dots \\ A^G / \operatorname{im} N & ; n = 2, 4, 6, \dots \end{cases}$$

■ We will be interested in the case that $A = k^\times$, the multiplicative group of a field k (which we will write multiplicatively, not additively as above), with trivial G-action on $k^\times$. The n-cycles for small n are

$$\begin{aligned} Z^0(G, k^\times) &= k^\times \\ Z^1(G, k^\times) &= \mathrm{Hom}_{\mathrm{grp}}(G, k^\times) = \big\{ \chi : G \to k^\times \,\big|\, \chi(gh) = \chi(g)\chi(h)\,,\ g, h \in G \big\} \\ Z^2(G, k^\times) &= \big\{ \varphi : G \times G \to k^\times \,\big|\, \varphi(b,c)\varphi(a,bc) = \varphi(ab,c)\varphi(a,b)\,,\ a,b,c \in G \big\} \\ Z^3(G, k^\times) &= \big\{ \omega : G^{\times 3} \to k^\times \,\big|\, \omega(b,c,d)\omega(a,bc,d)\omega(a,b,c) \\ &\qquad\qquad = \omega(ab,c,d)\omega(a,b,cd)\,,\ a,b,c,d \in G \big\} \end{aligned}$$

Exercise 2.38. 1. Let C_m be the cyclic group with m elements, with generator σ and group operation written multiplicatively, $\sigma^m = e$. Let ζ be an m'th root of unity in $k^\times$, that is, $\zeta^m = 1$ (ζ need not be primitive). Write $M = \{0, 1, 2, \dots, m-1\}$ so that each $g \in C_m$ can be uniquely written as $g = \sigma^a$ for some $a \in M$. Define $s : \mathbb{Z} \to \mathbb{Z}$ by the (E) property that for $a, b \in M$ also $a + b - ms(a + b)$ is in M. Explicitly $s(a + b) = 0$ for $a + b < m$ and $s(a + b) = 1$ for $a + b \geq m$. Set

$$\omega(\sigma^a, \sigma^b, \sigma^c) = \zeta^{a\, s(b+c)}$$

Show that ω is a normalised 3-cocycle. (For info: For $k = \mathbb{C}$ and $\zeta = \exp(2\pi i/m)$, the above ω is a generator of $H^3(C_m, \mathbb{C}^\times)$ in the sense that $H^3(C_m, \mathbb{C}^\times) = \{[\omega^a] \,|\, a = 0, 1, \dots, m-1\}$.)

2. The *group algebra* $k[G]$ for G over a field k has basis b_g, $g \in G$ and multiplication $b_g \cdot b_h = b_{gh}$, with unit b_e. For $\varphi : G \times G \to k^\times$ define a new product

$$b_g * b_h = \varphi(g, h)\, b_{gh}$$

Show that this defines an associative algebra with unit b_e if and only if φ is a normalised 2-cocycle. Denote the resulting k-algebra by $k[G]^\varphi$ – this is called a *twisted group algebra*.

Consider linear maps $f_\gamma : k[G] \to k[G]$, $b_g \mapsto \gamma(g) b_g$ for some $\gamma : G \to k^\times$. Show that there is an algebra isomorphism $k[G] \to k[G]^\varphi$ of this form if and only if $[\varphi] = 1$ (the trivial cohomology class) in $H^2(G, k^\times)$.

After these preparations, we turn to G-graded vector spaces Vec_G over a field k. For $A, B, C \in \mathsf{Vec}_G$ define a linear map $A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ as follows. Recall the notation $|a| \in G$ for the degree of a homogeneous element $a \in A$, etc. Let $\omega \in C^3(G, k^\times)$. We set

$$\begin{aligned} \alpha_{A,B,C} : A \otimes (B \otimes C) &\longrightarrow (A \otimes B) \otimes C \\ a \otimes (b \otimes c) &\longmapsto \omega(|a|, |b|, |c|)\, (a \otimes b) \otimes c \ , \end{aligned}$$

and we keep the unitors of Vec_G as they are:

$$\lambda_A : k_e \otimes A \to A, 1 \otimes a \mapsto a \quad , \quad \rho_A : A \otimes k_e \to A, a \otimes 1 \mapsto a \ .$$

Proposition 2.39. 1. Vec_G with associator and unitors as above is a monoidal category if and only if $\omega \in Z^3(G, k^\times)$ and ω is normalised. Denote the resulting monoidal category by Vec_G^ω.

2. For ω, ω' normalised 3-cocycles, the categories Vec_G^ω and $\mathsf{Vec}_G^{\omega'}$ are monoidally equivalent via a monoidal functor whose underlying functor is the identity functor if and only if $[\omega] = [\omega']$ in $H^3(G, k^\times)$.

Proof. For part 1 we need to check the axioms of a monoidal category in Definition 2.14. It is clear that $\alpha_{A,B,C}$ is an isomorphism, and that it is natural in A, B, C. (Why? Write out one of the naturality squares.) The ?

pentagon and triangle axioms, evaluated on homogeneous elements a, b, c, d with degrees $|a| = g_1$, $|b| = g_2$, etc, become

$$\omega(g_1, g_2g_3, g_4)\,\omega(g_2, g_3, g_4)\,\omega(g_1, g_2, g_3) \;=\; \omega(g_1, g_2, g_3g_4)\,\omega(g_1g_2, g_3, g_4)$$

and $\omega(g_1, e, g_2) = 1$. Together with Exercise 2.15, we see that these identities hold iff ω is a normalised 3-cocycle.

Part 2 is an exercise. □

Exercise 2.40. Show part 2 of the above proposition. That is, start from the identity functor $\mathrm{Id} : \mathsf{Vec}_G \to \mathsf{Vec}_G$ and try to upgrade it to a monoidal functor $(\mathrm{Id}, I^2, I^0) : \mathsf{Vec}_G^{\omega} \to \mathsf{Vec}_G^{\omega'}$. Convince yourself that up to monoidal natural isomorphism, one may assume $I_0 = 1_{\mathbb{1}} : \mathbb{1} \to \mathbb{1}$. E

Next we study braidings on Vec_G^{ω}, for $\omega \in Z^3(G, k^\times)$ normalised. Recall from Example 2.31 that these can only exist if G is abelian, so we assume this for the rest of this section. We make the ansatz

$$\beta_{A,B} : A \otimes B \to B \otimes A \quad , \quad a \otimes b \mapsto \psi(|a|, |b|)\, b \otimes a$$

for some function $\psi : G \times G \to k^\times$. Then β is natural by construction, and it remains to check the hexagon axioms from Definition 2.29. These read, for a, b, c homogeneous of degrees $|a| = g_1$, $|b| = g_2$, $|c| = g_3$,

$$\begin{aligned} \omega(g_3, g_1, g_2)\,\psi(g_1g_2, g_3)\,\omega(g_1, g_2, g_3) &= \psi(g_1, g_3)\,\omega(g_1, g_3, g_2)\,\psi(g_2, g_3)\ , \\ \omega(g_2, g_3, g_1)^{-1}\,\psi(g_1, g_2g_3)\,\omega(g_1, g_2, g_3)^{-1} &= \psi(g_1, g_3)\,\omega(g_2, g_1, g_3)^{-1}\,\psi(g_1, g_2)\ . \end{aligned}$$

(Check this.) One way to interpret this is that ω measures how far ψ is from being a bicharacter. A *bicharacter* on a group G with values in $k^\times$ is a map $b : G \times G \to k^\times$, such that, for all $g_1, g_2, h \in G$?

$$b(g_1g_2, h) = b(g_1, h)b(g_2, h) \quad , \quad b(h, g_1g_2) = b(h, g_1)b(h, g_2)\ .$$

Indeed, if $\omega = 1$, then ψ is a bicharacter.

■ A pair (ω, ψ) satisfying the above equations with $\omega \in Z^3(G, k^\times)$ is called an *abelian 3-cocycle* $Z^3_{\mathrm{ab}}(G, k^\times)$. It is called *normalised* if ω is normalised. By Exercise 2.30, this implies that ψ is automatically normalised as well. We denote the braided monoidal category obtained from a normalised abelian 3-cocycle by

$$\mathsf{Vec}_G^{(\omega,\psi)}\ .$$

For info: There is a cohomology theory adapted to abelian groups, which is different from general group cohomology, called *abelian group cohomology* H_{ab}. The higher cochains and cocycles get increasingly more complicated relative to group cohomology. One can identify both cohomology theories with cohomology of a topological space. Let $K(G, m)$ an Eilenberg-Maclane space, that is, a space with homotopy groups $\pi_m(K(G, m)) = G$ and $\pi_i(K(G, m)) = 1$ for $i \neq m$. Note that the only homotopy group which can be non-commutative is π_1, and so for $K(G, m)$ with $m \geq 2$, G needs to be commutative. $K(G, 1)$ is also called the *classifying space* BG of G. Then

$$H^n(G, \mathbb{Z}) = H^n(K(G, 1), \mathbb{Z}) \quad , \quad H^n_{\mathrm{ab}}(G, \mathbb{Z}) = H^n(K(G, 2), \mathbb{Z}) \ ,$$

with trivial action of G on $\mathbb{Z}$. See [Wei, Ex. 8.2.3], [JS2] and [Ga] for more on this.

■ Define the map $d_{\mathrm{ab}} : C^2(G, k^\times) \to Z^3_{\mathrm{ab}}(G, k^\times)$ as

$$d_{\mathrm{ab}}(\varphi) = (d\varphi, \varphi_{\mathrm{skew}}) \qquad \text{where} \quad \varphi_{\mathrm{skew}}(g, h) = \frac{\varphi(g, h)}{\varphi(h, g)} \ .$$

(Check that this is indeed an abelian 3-cocycle.) ?

Exercise 2.41. Show: For two normalised abelian 3-cocycles (ω, ψ) and (ω', ψ'), the corresponding braided monoidal categories $\mathsf{Vec}_G^{(\omega,\psi)}$ and $\mathsf{Vec}_G^{(\omega',\psi')}$ are braided monoidally equivalent with underlying functor the identity functor, if and only if they differ by $d_{\mathrm{ab}}(\varphi)$ for some $\varphi \in C^2(G, k^\times)$. Explicitly, E

$$\begin{aligned} \omega'(g_1, g_2, g_3) &= \omega(g_1, g_2, g_3)\, (d\varphi)(g_1, g_2, g_3) \ , \\ \psi'(g_1, g_2) &= \psi(g_1, g_2)\, \varphi_{\mathrm{skew}}(g_1, g_2) \ . \end{aligned}$$

■ The degree three abelian group cohomology with values in $k^\times$ is

$$H^3_{\mathrm{ab}}(G, k^\times) = Z^3_{\mathrm{ab}}(G, k^\times)/\mathrm{im}\, d_{\mathrm{ab}} \ .$$

There is a remarkable result relating $H^3_{\mathrm{ab}}(G, k^\times)$ to quadratic forms on G.

Definition 2.42. Let G be an abelian group. A *quadratic form* on G with values in $k^\times$ is a map $q : G \to k^\times$ such that $q(g) = q(g^{-1})$ and such that the function $b : G \times G \to k^\times$,

$$b(g, h) = \frac{q(gh)}{q(g)q(h)}$$

is a bicharacter on G. We write $QF(G, k^\times)$ for the quadratic forms on G.

Note that the product of two quadratic forms is again a quadratic form, and so $QF(G, k^\times)$ is an abelian group.

Theorem 2.43. Let G be an abelian group. The map

$$H^3_{\mathrm{ab}}(G, k^\times) \to QF(G, k^\times)$$

which takes $[\omega, \psi]$ to the quadratic form $q(g) = \psi(g, g)$ is well-defined and a group isomorphism.

We skip the proof, see [EGNO, Sec. 8.4] and [Ga, Thm. 4.4] for details.

■ The theorem makes the surprising statement that in order to characterise $\mathsf{Vec}_G^{(\omega,\psi)}$ up to braided monoidal equivalence (with underlying identity functor), all one needs to remember of associator and braiding is the self-braiding of the one-dimensional vector spaces k_g, concentrated in degree $g \in G$, for all g.

Exercise 2.44. In this exercise we will look at some properties of quadratic forms. Let G be an abelian group and $q \in QF(G, k^\times)$. Show: E

1. Let u be a bicharacter on G. Set $q(g) = u(g, g)$. Then q is a quadratic form. For such a quadratic form, $[1, u] \in H^3_{ab}(G, k^\times)$ gets mapped to q under the isomorphism of Theorem 2.43.

Let now $q \in QF(G, k^\times)$ be arbitrary.

2. $q(e) = 1$. *Hint:* A bicharacter is 1 if any of its arguments is the unit $e \in G$.

3. $q(g^a) = q(g)^{a^2}$ for $a \in \mathbb{Z}$. *Hint:* Evaluate $b(g, g^{-1})$ to see $b(g, g) = q(g)^2$, and proceed by induction.

4. Let now $G = C_m$, the cyclic group with m elements, and let $\sigma \in C_m$ be a generator. From part 3 we know $q(\sigma^a) = \zeta^{a^2}$ for some $\zeta \in k^\times$ with $\zeta^{m^2} = 1$. Show that in fact we need $\zeta^{2m} = 1$ for m even, and $\zeta^m = 1$ for m odd.

This shows that $QF(C_m, \mathbb{C}^\times) \cong \mathbb{Z}_{\gcd(m^2, 2m)}$.

5. Take now $k = \mathbb{C}$ and $m = 2$. By part 4 we have $QF(C_2, \mathbb{C}^\times) \cong \mathbb{Z}_4$. Write these four quadratic forms q_j, $j = 0, 1, 2, 3$, explicitly, and give abelian 3-cocycles (ω_j, ψ_j) which get mapped to q_j under the isomorphism of Theorem 2.43. For which j is $\mathsf{Vec}_{C_2}^{(\omega_j,\psi_j)}$ symmetric? Do you recognise these braided monoidal categories in our earlier examples?

2.6 Piecewise linear bordisms

There are many, many variants of bordism categories: topological, piecewise-linear, smooth, with various tangential structures, with embedded submanifolds (which can have their own tangential structures), extended versions which are higher bordism categories, ...

In this course, the focus is on algebraic aspects of TFTs, and we will not review or need the intricacies of the various notions. But to at least fix one setting in which all the considerations below work, we will quickly review *oriented piecewise linear bordisms*. See [RSa] for more on piecewise-linear structures.

Definition 2.45. 1. A map $f : \mathbb{R}^m \to \mathbb{R}^n$ is *piecewise-linear*, or PL, if there is a decomposition of $\mathbb{R}^m$ into a locally finite simplicial complex, so that f is affine-linear on each simplex.

2. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$. A map $f : U \to V$ is PL if it is the restriction of some PL map $\mathbb{R}^m \to \mathbb{R}^n$.

3. Let $n > 0$ and set

$$\mathbb{R}^n_\pm := \{(x_1, \dots, x_n) \in \mathbb{R}^n | \pm x_n \geq 0\} .$$

Let $U, V \subset \mathbb{R}^n_\pm$ be open. A *PL isomorphism* $U \to V$ is a bijective PL map. A PL isomorphism is *orientation preserving* if the determinant of the (linear part of the) affine linear map on each n-simplex is positive, and *orientation reversing* if it is negative.

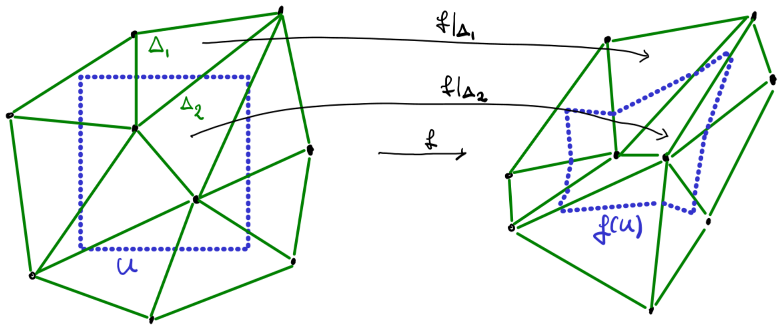
■ Let us expand on some points of this definition.

- A finite set $\{v_0, v_1, \dots, v_k\} \subset \mathbb{R}^m$ is called *independent* if it is not contained in an affine subspace of dimension $< k$, or, equivalently, if the k vectors $v_i - v_0$ are linearly independent. In this case, the k-simplex with vertices $\{v_0, \dots, v_k\}$ is the convex hull

$$\Delta(v_0, \dots, v_k) = \Big\{ \sum_{i=0}^{k} \lambda_i v_i \,\Big|\, \lambda_i \geq 0, \sum_i \lambda_i = 1 \Big\} .$$

An affine-linear map $f : \Delta(v_0, \dots, v_k) \to \mathbb{R}^n$ is uniquely determined by the images of the corner points $f(v_i)$, $i = 0, \dots, k$. (Why? Give the explicit linear map in terms of the $f(v_i)$. At which point are you using ?

independence of the set $\{v_i\}$? Give an example where $f(\Delta)$ is not a simplex.)



- The property that a PL map is determined on each simplex by the images of the corner points implies that a PL map is automatically continuous. (Why? *Hint:* Check that the restriction of the affine-linear maps from different simplices to a common face have to agree.) ?

- An affine linear map is of the form $f(x) = Ax + b$, where A is linear. Let us agree that $\det f := \det A$. For a PL isomorphism f from $U \subset \mathbb{R}^n$ to $V \subset \mathbb{R}^n$, we have that for each n-simplex Δ with $\Delta \cap U \neq \emptyset$ that $\det(f|_\Delta) \neq 0$.

- For $n \geq 2$ there is a map $R_{n-1,n} : \mathbb{R}^n_- \to \mathbb{R}^n_+$, given by

$$(x_1, \ldots, x_{n-2}, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-2}, -x_{n-1}, -x_n)$$

  is an orientation preserving PL isomorphism $\mathbb{R}^n_- \to \mathbb{R}^n_+$. For $n \geq 2$ it is thus enough to work with $\mathbb{R}^n_+$, even if we want to preserve orientation. However, including $\mathbb{R}^n_-$ allows for a more uniform treatment of the case $n = 1$ and is convenient for gluing later.

(15.4.26)

■ An *n-dimensional topological manifold* is a separable Hausdorff topological space[4] M such that each $p \in M$ has an open neighbourhood U homeomorphic to an open subset of $\mathbb{R}^n_\pm$. A pair (U, ϕ) of such a neighbourhood and homeomorphism $\phi : U \to \phi(U) \subset \mathbb{R}^n_\pm$ is called a *chart*. An *atlas* is a collection $\{(U_\alpha, \phi_\alpha)\}_\alpha$ of charts such that $\bigcup_\alpha U_\alpha = M$. The maps

$$\tau_{\beta,\alpha} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

are called *transition functions*. An atlas is *piecewise-linear* (or PL) if all transition functions are PL.

[4]separable = contains a countable dense subset; Hausdorff = distinct points have disjoint open neighbourhoods.

Definition 2.46. Let $n \geq 1$.

1. An *n-dimensional piecewise-linear (or PL) manifold* is an n-dimensional topological manifold with a PL atlas. It is *oriented* if the transition functions are orientation preserving.

2. Let M, N be PL manifolds, and let $V \subset M$. A *PL map* $V \to N$ is a map $f : V \to N$ such that f is PL in charts of M and N.

3. Let M be an n-dimensional oriented PL manifold with atlas $\{(U_\alpha, \phi_\alpha)\}_\alpha$. Write $R_n : \mathbb{R}^n_\pm \to \mathbb{R}^n_\mp$, $(x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1}, -x_n)$ for the reflection at the $x_n = 0$ plane. The *orientation reversal* $-M$ of M is M as a topological space, with atlas $(U_\alpha, R_n \circ \phi_\alpha)$.

(Check from the definition that the identity map from M to $-M$ is orientation reversing.) ?

■ The points of M which get mapped by a chart to

$$\mathbb{R}^{n-1} = \big\{(x_1, \ldots, x_{n-1}, 0)\big\} \subset \mathbb{R}^n_\pm$$

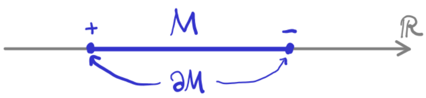
form an $(n-1)$-dimensional PL submanifold, called the *boundary* ∂M of M. If M is oriented, then ∂M is oriented, too. In a chart where a patch of the boundary ∂M gets mapped to $\mathbb{R}^{n-1} \subset \mathbb{R}^n_+$, the boundary is oriented as $\mathbb{R}^{n-1}$, and for $\mathbb{R}^{n-1} \subset \mathbb{R}^n_-$ it is oriented opposite to $\mathbb{R}^{n-1}$.

For example, take $n \geq 2$ and consider a chart $\phi : U \to \mathbb{R}^n_-$ containing part of the boundary of M. With the above convention, also $\phi' = R_{n-1,n} \circ \phi : U \to \mathbb{R}^n_+$ is a chart, and the map

$$R_{n-1,n} : \phi(U) \to \phi'(U)$$

is orientation preserving also on the boundary. (Why is it orientation preserving on the boundary?) ?

■ The case $n = 0$ needs a special treatment. A 0-dimensional PL manifold is the same as a 0-dimensional topological manifold (namely an at most countable number of isolated points, and finite in the compact case). An orientation on a 0-manifold M is an assignment of a sign $\pm$ to each point in M. A chart containing a boundary point of an oriented PL 1-manifold has values in exactly one of $\mathbb{R}_+$ or $\mathbb{R}_-$, and accordingly the boundary point is assigned orientation $+$ or $-$. For example:



■ From hereon we agree that:

- manifold = oriented PL manifold
- map between manifolds = PL map
- isomorphism between manifolds = orientation preserving PL isomorphism

■ *Bordisms:* A *closed* manifold is a compact manifold with empty boundary. Let X, Y be closed $(n-1)$-manifolds. A bordism

$$M : X \to Y$$

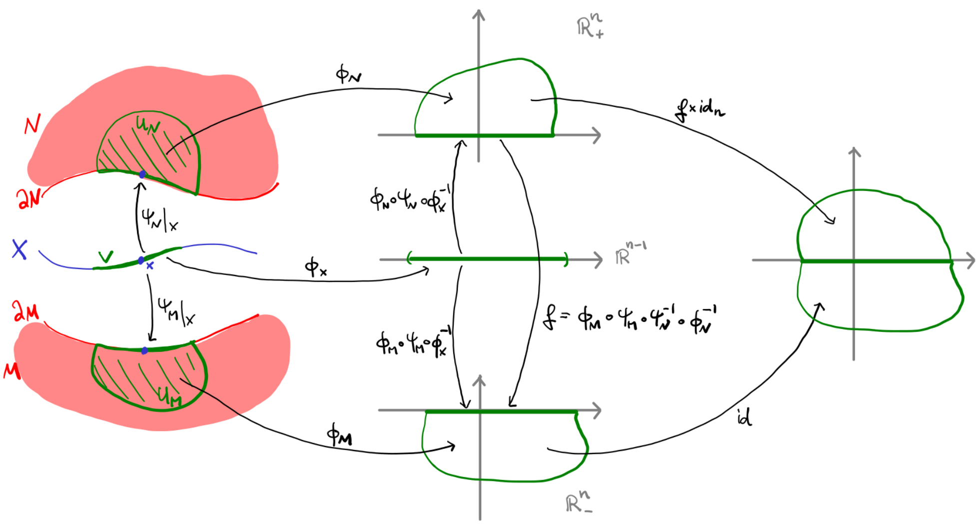
is an n-manifold M, together with an isomorphism $\psi_M : X \sqcup -Y \to \partial M$. Two bordisms $M, M' : X \to Y$ are *isomorphic* if there is an isomorphism $f : M \to M'$ compatible with the boundary parametrisations: $f \circ \psi_M = \psi_{M'} : X \sqcup -Y \to \partial M'$.

■ *Gluing:* Given bordisms $M : W \to X$ and $N : X \to Y$ we define a new bordism

$$N \sqcup_X M : W \to Y$$

by *gluing M and N along X.*

In more detail, for a point $x \in X$ pick neighbourhoods $V \subset X$ of x, $U_M \subset M$ of $\psi_M(x)$, $U_N \subset N$ of $\psi_N(x)$, such that $\psi_M(V) = U_M \cap \partial M$, $\psi_N(V) = U_N \cap \partial N$. By shrinking neighbourhoods, we may assume that we have charts $\phi_X : V \to \mathbb{R}^{n-1}$, $\phi_M : U_M \to \mathbb{R}^n_+$, $\phi_N : U_N \to \mathbb{R}^n_-$. We then build a chart around x in $N \sqcup_X M$ as in the figure below.



Alternatively one can work with collars, i.e. embeddings

$$c_N : X \times [0,1] \to N \quad , \quad c_M : X \times [-1,0] \to M \ ,$$

see [RSa, Cor. 2.26] for more on collars.

Definition 2.47. Let $n \geq 1$. The category Bord_n has

- objects: closed oriented $(n-1)$-manifolds.
- morphisms $X \to Y$: isomorphism classes $[M]$ of bordisms $M : X \to Y$.
- composition: For $[M] : X \to Y$ and $[N] : Y \to Z$, $[N] \circ [M]$ is the isomorphism class of $N \sqcup_Y M$.
- unit: 1_X is the isomorphism class of $X \times [0,1]$.

Bord_n becomes a monoidal category by taking the disjoint union as tensor product of objects and morphisms. It becomes symmetric via the swap of components $X \sqcup Y \to Y \sqcup X$.

■ Why PL and not e.g. smooth bordisms? This leads to a simpler and more combinatorial version of TFTs, which avoids some complications such as having to work with collars in the definition of gluing, or treating smooth structures on stratifications. (But it also limits the theory, for example it will never detect exotic smooth structures.)

Proposition 2.48. Bord_n is rigid. Specifically, $-X$ is a left and right dual of $X \in \mathsf{Bord}_n$.

Exercise 2.49. Show this. *Hint:* Consider $M = X \times [0,1]$ and interpret it as a bordism in different ways. E

We can now update our definition from the beginning:

Definition 2.50. An *n-topological field theory* (TFT for short) with values in a symmetric monoidal category S is a symmetric monoidal functor

$$Z : \mathsf{Bord}_n \to S \ .$$

We write

$$\mathsf{TFT}_n(S) := \mathrm{Fun}_{\mathrm{sym}}(\mathsf{Bord}_n, S)$$

for the category of n-dimensional TFTs with objects symmetric monoidal functors $\mathsf{Bord}_n \to S$ and morphisms monoidal natural transformations.

From this definition we immediately get the following corollary to Propositions 2.24 and 2.48.

Corollary 2.51. Let Z be an n-dimensional TFT. Then for every $X \in \mathsf{Bord}_n$, the object $Z(X) \in S$ has $Z(-X)$ as its left and right dual.

■ For example, if $S = \mathsf{Vec}_k$, this implies that $Z(X)$ is finite-dimensional for all X.

For info: From the motivation via path integrals in Section 1, we see that $Z(X) = \mathcal{H}(X)$ is interpreted as the state space of a quantum field theory on the $(n-1)$-manifold X. But QFTs typically have infinite-dimensional state spaces, and so TFTs model at best a very special type of QFT. One place where TFTs appear is in QFTs with an energy gap above the space of ground states, where TFTs model the behaviour of degenerate ground state spaces.

Lemma 2.52. Let C, D be monoidal categories, $F, G : C \to D$ be monoidal functors, and $\phi : F \Rightarrow G$ a monoidal natural transformation. If $a \in C$ has a left dual, then ϕ_a is invertible and ϕ_{a^*} is uniquely determined by ϕ_a. In particular, if C is rigid, then ϕ is invertible.

Exercise 2.53. Prove the above lemma in the case that C has left duals. Proceed as follows. By Theorem 2.27, it is enough to consider the case that C and D are strict. E

1. Let $a \in C$. By Proposition 2.24, $F(a)$ and $G(a)$ have duals. Show

$$(\phi_a \otimes \phi_{a^*}) \circ \mathrm{coev}_{F(a)} = \mathrm{coev}_{G(a)}$$

and the corresponding identity for evaluations.

2. Show that the morphism

$$\psi_a = \Big[G(a) \xrightarrow{\mathrm{coev}_{F(a)} \otimes 1_{G(a)}} F(a)F(a^*)G(a) \\ \xrightarrow{1_{F(a)} \otimes \phi_{a^*} \otimes 1_{G(a)}} F(a)G(a^*)G(a) \xrightarrow{1_{F(a)} \otimes \mathrm{ev}_{G(a)}} F(a) \Big]$$

is inverse to ϕ_a.

3. Use an expression similar to the one above to express ϕ_{a^*} in terms of $(\phi_a)^{-1}$.

■ Lemma 2.52 implies that $\mathsf{TFT}_n(S)$ is a groupoid. A *groupoid* is a category, all of whose morphisms are invertible.

Bibliographic notes for Section 2

The review of categorical notions and of the graphical calculus is all standard. Some textbook references are [Ka, ML, EGNO, TV]. For the strictification of monoidal categories, see [ML, Ch. XI.3] and [Ka, Ch. XI.5]. The validity of the graphical calculus for monoidal categories, including the braided case, is established in [JS1]. A detailed account of various graphical calculi in two dimensions can be found in [Sel]. The relation between associators and braidings on Vec_G and group cohomology goes back to [JS2]. A more detailed discussion of PL-bordisms will be given in [CM]. A discussion of bordism categories and topological field theory can also be found in [Ko, Sec. 1.2] or [TV, Sec. 10.2], including the dualisability property in Corollary 2.51. It is also reviewed in [CR2].

3 Dimension one

3.1 Generators for monoidal categories

■ We need to upgrade our strictification result to monoidal functors. A monoidal functor F is called *strict* if F^2 and F^0 are identities.

Let C, D be monoidal categories, and $F : C \to D$ a monoidal functor. We will construct a strict monoidal functor F^{str} such that the following diagram commutes via a monoidal natural isomorphism η:

$$\begin{tikzcd} C^{\mathrm{str}} \arrow[r, "F^{\mathrm{str}}"] \arrow[d, "T^C"'] & D^{\mathrm{str}} \arrow[d, "T^D"] \arrow[dl, phantom, "\swarrow\eta"] \\ C \arrow[r, "F"'] & D \end{tikzcd}$$

For $\underline{a} = (a_1, \dots, a_m)$ in C^{str} we set

$$F^{\mathrm{str}}(\underline{a}) = F(a_1) * \cdots * F(a_m) \quad \text{and} \quad F^{\mathrm{str}}(()) = () \ .$$

Then

$$T^D \circ F^{\mathrm{str}}(\underline{a}) = Fa_1 \otimes (Fa_2 \otimes \cdots \otimes (Fa_{m-1} \otimes Fa_m) \cdots) \ ,$$
$$F \circ T^C(\underline{a}) = F\big(a_1 \otimes (a_2 \otimes \cdots \otimes (a_{m-1} \otimes a_m) \cdots) \big) \ .$$

We can use an iterated application of $F^2 : F(u) \otimes F(v) \to F(u \otimes v)$ to get an isomorphism

$$\eta_{\underline{a}} : T^D \circ F^{\mathrm{str}}(\underline{a}) \to F \circ T^C(\underline{a}) \ .$$

If $\underline{a} = ()$, then $\eta_{()} : \mathbb{1} \to F(\mathbb{1})$ is given by F^0.

For a morphism $f : \underline{a} \to \underline{b}$ in C^{str} we define

$$F^{\mathrm{str}}(f) = \Big[T^D(F^{\mathrm{str}}(\underline{a})) \xrightarrow{\eta_{\underline{a}}} F(T^C(\underline{a})) \xrightarrow{F(f)} F(T^C(\underline{b})) \xrightarrow{\eta_{\underline{b}}^{-1}} T^D(F^{\mathrm{str}}(\underline{b})) \Big]$$

Proposition 3.1. F^{str} is a strict monoidal functor and $\eta : T^D \circ F^{\mathrm{str}} \Rightarrow F \circ T^C$ is a monoidal natural isomorphism.

Exercise 3.2. Prove (bits of) the proposition. Namely, show that (i) F^{str} is a functor, (ii) F^{str} is strict monoidal (i.e. it makes sense to set the coherence isomorphisms of the monoidal functor to 1 as the corresponding objects are equal), (iii) η is a natural isomorphism. The more tedious part is to show that η is monoidal. This you do not have to do, but at least (iv): write down the commuting square that is required. And maybe think how you would set up an inductive argument. E

■ Let C be a monoidal category and C^{str} its strictification, with monoidal equivalence $T : C^{\mathrm{str}} \to C$ as in Section 2.2.

- Let C_0 be a subset of objects in C. Then C_0 defines a full monoidal subcategory C_0^{str} of C^{str} consisting of lists $\underline{a} = (a_1, \dots, a_m)$ with $a_i \in C_0$ (and of all morphisms between these objects).
- Let C_1 be a subset of morphisms in C_0^{str}. Thus an element $g \in C_1$ is a morphism $g : T(\underline{a}) \to T(\underline{b})$ in C with $\underline{a}, \underline{b} \in C_0^{\mathrm{str}}$.

We say that C is *tensor generated* by a set of objects C_0 and a set of morphisms C_1 if:

1. Every object in C is isomorphic to $T(\underline{a})$ for some $\underline{a} \in C_0^{\mathrm{str}}$. The empty list is allowed and $T(()) = \mathbb{1}$.
2. For $\underline{a}, \underline{b} \in C_0^{\mathrm{str}}$, every morphism $f : T(\underline{a}) \to T(\underline{b})$ in C can be written as
$$f = \left[\underline{a} \xrightarrow{h_1} \underline{c}_1 \xrightarrow{h_2} \cdots \xrightarrow{h_{n-1}} \underline{c}_{n-1} \xrightarrow{h_n} \underline{b}\right] ,$$
for some $\underline{c}_i \in C_0^{\mathrm{str}}$, and where each h_i is a tensor product in C_0^{str} of identity morphisms and morphisms in C_1.

(21.4.26)

Point 2 states, equivalently, that every morphism in C_0^{str} can be written as a composition of tensor products of morphisms in C_1 and identities. Write

$$I : C_0^{\mathrm{str}} \hookrightarrow C^{\mathrm{str}}$$

for the embedding of the subcategory C_0^{str}. Clearly, I is a strict monoidal functor. Point 1 implies that I is an equivalence. (Why is I fully faithful and essentially surjective?) ?

Lemma 3.3. Let C be tensor generated by C_0, C_1, and let D be a monoidal category. A monoidal functor $C \to D$ is uniquely determined, up to natural monoidal isomorphism, by its values on C_0 and C_1.

Proof. Let $F, G : C \to D$ be monoidal functors, and let $F^{\mathrm{str}}, G^{\mathrm{str}} : C^{\mathrm{str}} \to D^{\mathrm{str}}$ be their strictifications.

Suppose that F and G agree on C_0 in the sense that $F(a) = G(a)$ for all $a \in C_0$. Then also $F^{\mathrm{str}}(\underline{a}) = G^{\mathrm{str}}(\underline{a})$ for all $\underline{a} \in C_0^{\mathrm{str}}$.

Suppose further that F and G agree on C_1 in the sense that $F^{\mathrm{str}}(g) = G^{\mathrm{str}}(g)$ for all $g \in C_1$.

As all morphisms in C_0^{str} can be written as compositions and tensor products of those in C_1 (and identities), and as $F^{\mathrm{str}}, G^{\mathrm{str}}$ are strict monoidal, we have
$$F^{\mathrm{str}} \circ I = G^{\mathrm{str}} \circ I \ : \ C_0^{\mathrm{str}} \to D^{\mathrm{str}} ,$$

where $I : C_0^{\mathrm{str}} \hookrightarrow C^{\mathrm{str}}$ is the embedding from above. Write $\tilde{I}, \tilde{T^C}$ for the inverse part of the equivalence. Then

$$F \cong \Big[C \xrightarrow{\tilde{T^C}} C^{\mathrm{str}} \xrightarrow{\tilde{I}} \underbrace{C_0^{\mathrm{str}} \xrightarrow{I} C^{\mathrm{str}} \xrightarrow{F^{\mathrm{str}}} D^{\mathrm{str}}}_{=G^{\mathrm{str}} \circ I} \xrightarrow{T^D} D \Big] \cong G$$

as monoidal functors. □

Lemma 3.4. Let C be tensor generated by C_0, C_1, let D be a monoidal category, and $F, G : C \to D$ functors. Then

1. a monoidal natural transformation $F \Rightarrow G$ is uniquely determined by its values on C_0, and

2. given a collection of morphisms $(f_a : F(a) \to G(a))_{a \in C_0}$, there is a natural monoidal transformation $\phi : F \to G$ with $\phi_a = f_a$ for all $a \in C_0$ if and only if the (f_a) are compatible with C_1 in the following sense: for $\underline{a} \in C_0^{\mathrm{str}}$ set $f_{\underline{a}} = f_{a_1} * \cdots * f_{a_m}$; we require that for all $g : \underline{a} \to \underline{b}$ in C_1:

$$\begin{tikzcd} F^{\mathrm{str}}(\underline{a}) \arrow[r, "F^{\mathrm{str}}(g)"] \arrow[d, "f_{\underline{a}}"] & F^{\mathrm{str}}(\underline{b}) \arrow[d, "f_{\underline{b}}"] \\ G^{\mathrm{str}}(\underline{a}) \arrow[r, "G^{\mathrm{str}}(g)"] & G^{\mathrm{str}}(\underline{b}) \end{tikzcd}$$

Exercise 3.5. Show this. *Hint:* First check that it is enough to show the existence and uniqueness statement for natural monoidal transformations $F^{\mathrm{str}} \circ I \Rightarrow G^{\mathrm{str}} \circ I$ for $I : C_0^{\mathrm{str}} \hookrightarrow C^{\mathrm{str}}$ the embedding, cf. Lemma 2.10. Then use strictness of the monoidal functors and the generating property of C_1. E

■ Denote by $\bullet_\pm$ a point with orientation $+$ or $-$. We will import the following result, which can be shown for example by Morse theory.

Proposition 3.6. Bord_1 is tensor generated by $C_0 = \{\bullet_+, \bullet_-\}$ and

$C_1 = \Big\{$  $\Big\}$

(Why can one not leave out, say, the right evaluation and coevaluation and just have three generators? How does one obtain the symmetric braiding for all four combinations $\bullet_\pm \sqcup \bullet_\pm$?) ?

■ For a monoidal category C, denote by C^{fd} ("fully dualisable") the full subcategory of objects which have left and right duals. This is in fact a monoidal subcategory. (Why? And is it not confusing that "fd" could also stand for "finite-dimensional"?)

Furthermore, for an arbitrary category D, denote by $D^\times$ the maximal subgroupoid of D, that is the subcategory consisting of all objects of D, but only of invertible morphisms. If D is monoidal, then $D^\times$ is a monoidal subcategory. (Why?)

Below, for a monoidal category C, we are interested in the groupoid of dualisable objects, $(C^{\mathrm{fd}})^\times$. (Why is $(C^{\mathrm{fd}})^\times$ in general not the same as $(C^\times)^{\mathrm{fd}}$? What would the second category be for $C = \mathsf{Vec}_k$?)

Theorem 3.7. Let S be a symmetric monoidal category. Then $\mathsf{TFT}_1(S) \cong (S^{\mathrm{fd}})^\times$ as categories via

$$Z \mapsto Z(\bullet_+) \quad \text{and} \quad (\phi : Y \Rightarrow Z) \;\mapsto\; (\phi_{\bullet_+} : Y(\bullet_+) \to Z(\bullet_+)) \ .$$

Proof (sketch). The assignment is well-defined and a functor. Indeed, $Z(\bullet_+)$ is dualisable in S by Corollary 2.51, and $\phi_{\bullet_+}$ is invertible by Lemma 2.52. (Why is it a functor? I.e. why is it compatible with composition and preserves identities?)

Faithful: By Lemma 3.4, a natural transformation is determined by its value on $\bullet_+$ and $\bullet_-$. By Lemma 2.52, $\phi_{\bullet_-}$ is determined by $\phi_{\bullet_+}$. So $\phi = \psi$ iff $\phi_{\bullet_+} = \psi_{\bullet_+}$.

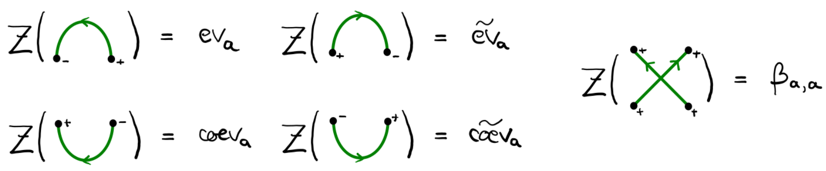
Full: Let $Y, Z \in \mathsf{TFT}_1(S)$ and $f : Y(\bullet_+) \to Z(\bullet_+)$ be an isomorphism. We set $\phi_{\bullet_+} = f$ and take $\phi_{\bullet_-}$ the corresponding unique value determined by Lemma 2.52. By Lemma 3.4, we need to check this assignment is compatible with the generating morphisms. For the (co)evaluation in Bord_1, this follows by construction of $\phi_{\bullet_-}$, and for the braiding it is naturality of the braiding in S. (Fill in some of the details.)

Essentially surjective (sketch): We claim that every dualisable object in S can occur as the value of $Z(\bullet_+)$. So, let $a \in S^{\mathrm{fd}}$ be given.

We assign the following values to the generating sets C_0 and C_1:

$$Z(\bullet_+) = a \quad , \quad Z(\bullet_-) = a^* \ ,$$

and



Let $M : X \to Y$ be a bordism in Bord_1. By Proposition 3.6 it can be written in terms of the generating morphisms in C_1. This determines the value of Z on M.

Now comes the part where the proof becomes a sketch: It remains to show that two different ways of writing isomorphic bordisms in terms of the generators result in the same value for Z. This can again be done via Morse theory, which shows that the zig-zag is the only relation to check, and from the fact that the exchange of points gets mapped to a symmetric braiding, which shows independence of how level sets of the Morse function are mapped to an ordered set of oriented points. $\square$

For info: The simple description of one-dimensional TFTs as the groupoid of dualisable objects will not persist in higher dimensions, at least in our 1-categorical setting. However, remarkably, it does stay true in all dimensions if one works with fully extended topological field theories, where the source category is the symmetric monoidal (∞, n)-category of framed n-bordisms, and the target is any symmetric monoidal (∞, n)-category. In that setting, giving an n-dimensional TFT is the same as giving a fully dualisable object in the target category. This is the content of Lurie's proof outline of the Baez-Dolan cobordism hypothesis, see [Lu1].

3.2 Stratifications and colourings

Now we make the bordism category more complicated by adding stratifications and colourings (or labellings, or decorations).

■ Let P be a partially ordered set, or poset for short. P can be made a topological space by declaring that a set is open if it is closed upwards:

$$U \subset P \text{ open} \quad \overset{\text{Def.}}{\Leftrightarrow} \quad \text{if } p \in P,\, u \in U \text{ with } p \geq u, \text{ then also } p \in U.$$

Exercise 3.8. (E)

1. Show that (i) a map between posets is continuous if and only if it is order preserving, and (ii) the sets $\{P_{\geq x}\}_{x \in P}$ are a base of the topology on P, that is, every open set $U \subset P$ can be written as a union of sets $P_{\geq x} := \{p \in P \,|\, p \geq x\}$.

2. Show that for $x \in P$, the set $P_{<x} = \{p \in P \,|\, p < x\}$ is closed. Give an example where the complement of $P_{<x}$ is not $P_{\geq x}$. Give an example of a totally ordered set where each one-point set is neither open nor closed.

Definition 3.9. Let P be a poset and X a topological space. A *P-stratification of X* is a continuous map $s : X \to P$.

■ We write $X_{\geq p}$ for all $x \in X$ with $s(x) \geq p$, and analogously for $X_{>p}$, $X_{\leq p}$, $X_{<p}$. Instead of $X_{=p}$ we will write X_p. Note that $X_{\geq p} = s^{-1}(P_{\geq p})$, and so these sets are open in X. Analogously, by the above exercise $X_{<p} = s^{-1}(P_{<p})$ is a closed subset of X. (Actually, also $X_{>p}$ is open and $X_{\leq p}$ is closed. Check this.) ?

Example 3.10. 1. Take $P = \{0, 1, 2\}$ with induced total order. The open sets are $\emptyset$, $\{2\}$, $\{1, 2\}$, $\{0, 1, 2\}$.

Then the map $\mathbb{R}^2 \to P$, $x \mapsto$ (number of coordinates of x that are non-zero) is a stratification. $(\mathbb{R}^2)_{\geq 2}$ is $\mathbb{R}^2$ minus its coordinate axes, and $(\mathbb{R}^2)_{\geq 1}$ is $\mathbb{R}^2 \setminus \{0\}$. (What is $(\mathbb{R}^2)_1$? What is the generalisation to $\mathbb{R}^n$? What is $(\mathbb{R}^3)_{\geq 2}$?) ?

On the other hand, the map which takes x to the number of coordinates that are zero is not a stratification of $\mathbb{R}^2$.

2. Take $P = \{0, +, -\}$ with $0 < +$, $0 < -$ and no relation between $+, -$. The sign map $\mathrm{sgn} : \mathbb{R} \to P$, which is $+, -, 0$ depending on whether x is positive, negative, or zero, is a stratification. More generally, a P-stratification of $\mathbb{R}$ is the same as giving two disjoint open sets $U_\pm \subset \mathbb{R}$. (Why?) On the other hand, if we take $Q = \{+, -\}$ with no relation (discrete topology), there is no Q-stratification of $\mathbb{R}$ which has both $+$ and $-$ in its image. (Why?) ? ?

(22.4.26)

■ At this point, stratified spaces are still too general. We would like to make them interact well with PL-manifolds.

Definition 3.11. Let $\{0, 1, \ldots, n\}$ have induced total order, let D be a poset and $\pi : D \to \{0, 1, \ldots, n\}$ a strictly increasing map. A *D-stratified piecewise-linear n-manifold* is a PL n-manifold M together with a stratification $s : M \to D$. Denote the preimage of $d \in D$ by

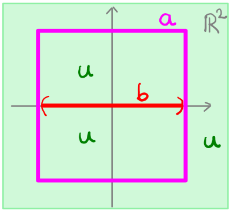
$$M_d := s^{-1}(\{d\}) \ .$$

We demand that M_d is a (PL) submanifold of dimension $\pi(d)$ satisfying $\partial M_d = M_d \cap \partial M$. A connected component of $(\pi \circ s)^{-1}(\{k\})$ is called a *k-stratum*. We denote the elements in D that can label a k-stratum by

$$D_k = \pi^{-1}(\{k\}) \ .$$

■ That π is strictly increasing means that $a < b$ implies $\pi(a) < \pi(b)$. The reason to demand this is so that closures of different strata can only intersect in lower dimensional strata.

Here is an example of what can go wrong otherwise: Let $D = \{x, a, b, u\}$ with $\pi : D \to \{0, 1, 2\}$, $\pi(x) = 0$, $\pi(a) = \pi(b) = 1$, $\pi(u) = 2$. Introduce two different partial orders: D^α has $x < a < b < u$, and D^β has $x < a < u$, $x < b < u$ with no relation between a, b. Then π is strictly order preserving for D^β, but not for D^α. Consider the decomposition of $\mathbb{R}^2$ into the unit square, the open interval $(-1, 1)$ on the x-axis, and the complement of all that, with map $s : \mathbb{R}^2 \to D$ as follows:



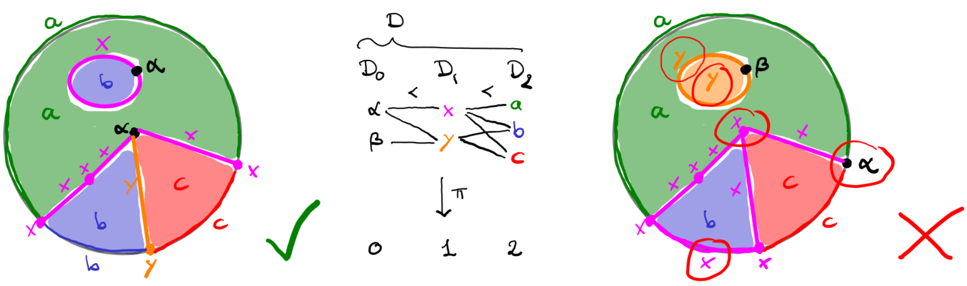
Then s is a stratification for D^α but not for D^β. Indeed, the preimage of $D^\alpha_{\geq a} = \{a, b, u\}$ is $\mathbb{R}^2$, while the preimage of $D^\beta_{\geq a} = \{a, u\}$ is $\mathbb{R}^2$ less the b-labelled open interval, and is not open. (Check also the preimages of the open sets for $\geq x$, $\geq b$.) If we take s to be x on the points ± 1 on the x-axis, then we get a stratification for both D^α and D^β. ?

Lemma 3.12. Let x_n be a sequence of points in M converging to some $x \in M$. Suppose $s(x_n) = b$ and $s(x) = a$. Then $a \leq b$.

Proof. Suppose $a \leq b$ does not hold. Then $b \notin D_{\geq a}$ and so $x_n \notin s^{-1}(D_{\geq a})$. But $x \in s^{-1}(D_{\geq a})$ and $s^{-1}(D_{\geq a})$ is open. So x has a neighbourhood not containing points of the sequence x_n, in contradiction to $x_n \to x$. $\square$

In particular, if $a \neq b$, then $a < b$ and hence also $\pi(a) < \pi(b)$. That is, the dimension of M_a is smaller than that of M_b.

Example 3.13. On the left an example is shown which is a D-stratified 2-manifold M, and on the right we have an example which is not:



The problematic parts on the right are circled in red: y labels a 1-stratum adjacent to a 2-stratum labelled a, but there is no relation $y < a$, and so

$M_{\geq y}$ is not open; y also labels a 2d region, so M_1 is not a 1-manifold; x labels a vertex where three x-labelled 1-strata meet, and so again M_1 is not a 1-manifold; there is a 0-stratum labelled α on the boundary, so that $M_0 \cap \partial M \neq \emptyset$, but $\partial M_0 = \emptyset$; there is a 1-stratum labelled x on the boundary, and so $\partial M_1 \neq M_1 \cap \partial M$.

■ The definition implies many other properties, for example:

- There is a simplicial decomposition of M such that the value of s is constant in the interior Δ° of each k-simplex ($k = 1, \dots, n$).

  Indeed, M_d is a k-manifold, for $k = \pi(d)$, and so has a PL-atlas (compatible with that of M).

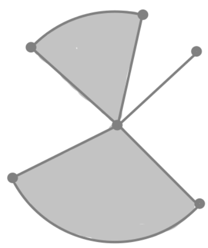
- Choose a simplicial decomposition as in the first point and let $x \in M$ be in the interior of a k-simplex. Then $\pi(s(x)) \geq k$, and $\geq k+1$ if $x \in \partial M$.

  Write $d = s(x)$ and $\ell = \pi(d)$. Then M_d is an ℓ-manifold by assumption. We cannot have $\ell < k$ as an ℓ-manifold cannot contain a k-simplex. If $x \in \partial M$, then also $x \in \partial M_d$. As ∂M_d is a $(\ell-1)$-manifold we get $k \leq \ell - 1$.

- s is constant on a k-stratum. (Why? *Hint:* Let S be the k-stratum. Let $u, v \in D$ satisfy $\pi(u) = k = \pi(v)$. Then $v \notin M_{\geq u}$ and $u \notin M_{\geq v}$. Check that the intersections of $M_{\geq u}$ and $M_{\geq v}$ with S are open in S and disjoint. Then use that S is connected.) ?

- For info: *Frontier condition:* Let $A \subset M_a$, $B \subset M_b$ be strata (i.e. connected components) for $a, b \in D$, and denote by $\overline{A}$ the (topological) closure of A. Then $\overline{A} \cap B \neq \emptyset$ implies $B \subset \overline{A}$.

For info: There are more flexible ways to restrict general stratified spaces to something workable. One example are so-called *conically stratified spaces*, which are inductively modeled on $C(Z) \times \mathbb{R}^k$ where Z is stratified and compact of lower dimension. See [Lu2, App. A.5] and [AFT, Sec. 2.1] for more on this. This allows spaces which do not have an underlying manifold, for example they could locally look like



- Write $D_{\geq 1}$ for the preimage under π of $\{1, \dots, n\}$ and define

$$\pi_{\geq 1} : D_{\geq 1} \to \{0, 1, \dots, n-1\} \quad , \quad x \mapsto \pi(x) - 1 \ .$$

A *D-stratified n-dimensional bordism* is a bordism $M : X \to Y$, together with stratifications $s_M : M \to D$, $s_X : X \to D_{\geq 1}$, $s_Y : Y \to D_{\geq 1}$ compatible with the boundary parametrisation $\psi_M : X \sqcup -Y \to \partial M$.

We will not actually be working with general stratified bordisms, and will use a more restrictive notion where we give explicit local models. We will refer to these as *defect-bordisms*.

- *1-dimensional defect bordisms:* Let $D = D_0 \sqcup D_1$ be given, with maps $s, t : D_0 \to D_1$. Turn D into a poset by declaring $x < s(x)$ and $x < t(x)$ for all $x \in D_0$. Let $\pi : D \to \{0, 1\}$ map D_i to i.

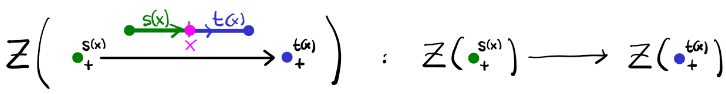
Definition 3.14. A 1-*dimensional defect bordism* for D is a D-stratified bordism $M : X \to Y$, such that each point of M has a neighbourhood isomorphic to one of the following, for $x \in D_0$, $a \in D_1$:

The category $\mathsf{Bord}_1^{\mathrm{def}}(D)$ has as objects $D_{\geq 1}$ stratified points and as morphisms isomorphism classes of 1-dimensional defect bordisms for D.

- Accordingly, a *1-dimensional defect TFT for D* with values in a symmetric monoidal category S is a symmetric monoidal functor

$$Z : \mathsf{Bord}_1^{\mathrm{def}}(D) \to S \ .$$

Example 3.15. Let us take $S = \mathsf{Vec}_k$. Then Z assigns to each $a \in D_1$ a finite-dimensional vector space V_a via $V_a = Z(\bullet_+^a)$, with $\bullet_+^a$ the a-labelled positively oriented point. To each $x \in D_0$, Z assigns a linear map $f_x : V_{s(x)} \to V_{t(x)}$ via



(Consider the reflected version of the picture above as a morphism $\bullet_-^{t(x)} \to \bullet_-^{s(x)}$. What is the corresponding linear map?) ?

3.3 State sum models

Take $D_0 = D_1 = \{*\}$ to be one-element sets, so there is a unique colour for 0-strata and 1-strata. The defect TFT $Z : \mathsf{Bord}_1^{\mathrm{def}}(D) \to S$ is determined by the object $a \in S$ assigned to $\bullet_+$ and the morphism $f : a \to a$ assigned to the bordism I_1 consisting of an interval with one zero-stratum. Let I_n be the interval with n 0-strata. Then

$$Z(I_n) = Z\big(\underbrace{I_1 \circ \cdots \circ I_1}_{n \text{ times}}\big) = \underbrace{Z(I_1) \circ \cdots \circ Z(I_1)}_{n \text{ times}} = f^n : a \to a$$

is the n-fold composition of f with itself.

Example 3.16. *The Ising lattice model.* The n-dimensional Ising lattice model is defined on an n-dimensional square lattice $\mathbb{Z}^n$ (or some subset Λ thereof), with a degree of freedom $\{\pm 1\}$ – a "spin" – at each site. A configuration is therefore a map $\sigma : \Lambda \to \{\pm 1\}$. The energy of each configuration is

$$E(\sigma) = -\sum_{\langle ij \rangle} \sigma_i \sigma_j \ ,$$

where the sum is over pairs $\langle ij \rangle$ of neighbouring sites, and σ_i is the value of σ at $i \in \Lambda \subset \mathbb{Z}^n$. Here we take Λ finite to make $E(\sigma)$ a finite sum. The partition function on Λ is

$$Z_\Lambda(\beta) = \sum_\sigma e^{-\beta E(\sigma)} \ ,$$

where the sum is over all configurations $\sigma : \Lambda \to \{\pm 1\}$.

We want to use 1d defect TFT to build the 1d Ising lattice model. Take $D_0 = D_1 = \{*\}$ as above, $S = \mathsf{Vec}_{\mathbb{C}}$, and consider the defect TFT with $Z(\bullet_+) = \mathbb{C}^2$ and

$$Z(I_1) = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}$$

Then $Z(I_n)$ computes the Ising model partition function with n sites, except that the TFT bordism is Poincaré-dual to the Ising lattice in that the degrees of freedoms $\{\pm 1\}$ are associated to the 1-strata, and the interaction to the 0-strata.

To see this, label the standard basis of $\mathbb{C}^2$ as $e_+ = (1,0)$ and $e_- = (0,1)$. Write $e_\pm^*$ for the corresponding dual basis of $(\mathbb{C}^2)^*$. Then for a lattice with n sites and spin degrees of freedom on the two boundaries fixed to be "+", we compute

$$e_+^* Z(I_n) e_+ = e_+^* Z(I_1) \circ Z(I_1) \circ \cdots \circ Z(I_1) e_+$$

$$= \sum_{\sigma_1,\dots,\sigma_{n-1}=\pm 1} e_+^* Z(I_1) e_{\sigma_1}\, e_{\sigma_1}^* Z(I_1) e_{\sigma_2} \cdots e_{\sigma_{n-1}}^* Z(I_1) e_+$$

$$= \sum_{\sigma_1,\dots,\sigma_{n-1}=\pm 1} e^{\beta\left(\sigma_1 + \sum_{k=1}^{n-2} \sigma_k\sigma_{k+1} + \sigma_{n-1}\right)} = \sum_{\sigma} e^{-\beta E(\sigma)} \ ,$$

where in the last sum, we fix $\sigma_0 = + = \sigma_n$. Note that the Ising model is not topological and depends on the lattice size via n in I_n. Still, we have used TFT to make a non-topological lattice model functorial: $Z(I_m \circ I_n) = Z(I_m) \circ Z(I_n)$. This is a discrete (and rigorous) instance of the idea in Section 1 that the path integral should be modeled by a functor. For more on the 1d Ising model, see [Mus, Sec. 2.1].

Exercise 3.17. Compute $e_+^* Z(I_n) e_+$ as a function of n in terms of $\cosh\beta$ and $\sinh\beta$ by diagonalising $Z(I_1)$. If you know what is meant by "phase transition", consider $e_+^* Z(I_n) e_+ / \big((e_+^* + e_-^*) Z(I_n)(e_+ + e_-)\big)$ and argue that the 1d Ising model has no phase transition between an ordered and disordered phase. Ⓔ

■ *New 1d TFTs from state sums.* Let us return to a general symmetric monoidal S. We can ask if a state sum model as above can also be topological. In view of the identity

$$Z(I_m \circ I_n) = Z(I_m) \circ Z(I_n)$$

from above, we would certainly need that $Z(I_n)$ is independent of n, which in turn follows if

$$Z(I_2) = Z(I_1) \quad \Leftrightarrow \quad f \circ f = f : a \to a \ .$$

In other words, $Z(I_n)$ is independent of n ($n \geq 1$) iff $f \in \mathrm{End}(a)$ is an idempotent. We can try to define a new TFT $Z^{\mathrm{new}} : \mathsf{Bord}_1 \to S$ (without defects for now) by setting

$$Z^{\mathrm{new}}(I) = Z(I_n) \quad \text{for any } n > 0 \ .$$

However, that is not yet a TFT because the (unstratified) interval I is the identity on $\bullet_+$ in Bord_1, and it should be mapped to the identity 1_a on a in S. But it does get mapped to f, which is just an idempotent on a and not necessarily equal to 1_a. We need one more ingredient:

Definition 3.18. Let C be a category and $a \in C$. A morphism $p : a \to a$ is called an *idempotent* if $p \circ p = p$. An idempotent p is *split* if there are

$$j \in C \quad , \quad e : j \to a \ , \quad r : a \to j \ , \quad re = 1_j \ , \quad er = p \ .$$

In this case, $j = (j, e, r)$ is called a *splitting of* p, or the *image of* p. If each idempotent in C is split, C is called *idempotent complete.* (28.4.26)

Exercise 3.19. Let C be a category. Ⓔ

1. A morphism $m : x \to y$ is called *mono* or a *monomorphism* (think "injective map") if it is *left-cancellable:* if $f, g : u \to x$ satisfy $m \circ f = m \circ g$, we have $f = g$. A morphism $e : x \to y$ is called *epi* or a *epimorphism* (think "surjective map") if it is *right-cancellable:* if $f, g : y \to v$ satisfy $f \circ e = g \circ e$, we have $f = g$.

   Let (j, e, r) be a splitting of an idempotent $p : a \to a$. Show that e is mono (e for "embedding", not for "epi") and r ("restriction") is epi. Show that $pe = e$, $rp = r$. Describe in which sense a splitting of an idempotent unique up to unique isomorphism.

2. *Idempotent completion, or Karoubi envelope.* We would like to construct a category $\mathrm{Kar}(C)$ from C in which all idempotents split, and which is in some sense canonical with that property. We define $\mathrm{Kar}(C)$ to have

   - objects: pairs (a, p) where $a \in C$ and $p : a \to a$ is an idempotent.
   - morphisms $(a, p) \to (b, q)$: morphisms $f : a \to b$ in C which satisfy $qfp = f$.

   Show that $\mathrm{Kar}(C)$ is indeed a category, and that it is idempotent complete.

3. Show that the embedding $I : C \to \mathrm{Kar}(C)$, $a \mapsto (a, 1_a)$ and $f \mapsto f$ is fully faithful. Show that if C is already idempotent complete, then I is in addition essentially surjective, and hence an equivalence. Conclude that $\mathrm{Kar}(\mathrm{Kar}(C)) \cong \mathrm{Kar}(C)$, so that the term "completion" makes sense.

For info: If C' is another idempotent complete category with a fully faithful embedding $I' : C \to C'$, and if every object in C' is isomorphic to the splitting of some idempotent in C, then C' is equivalent to $\mathrm{Kar}(C)$.

■ Suppose S is idempotent complete, and we are given a defect TFT $Z : \mathsf{Bord}_1^{\mathrm{def}}(D) \to S$ with $D_0 = D_1 = \{*\}$. Let $a = Z(\bullet_+)$ and $p = Z(I_1)$, and suppose that p is an idempotent. Let (j, e, r) be a splitting of p. We get our new (unstratified) TFT by setting, $Z^{\mathrm{new}}(\bullet_+) = j$ and, for any $n \geq 1$,

$$Z^{\mathrm{new}}(I) = \left[Z^{\mathrm{new}}(\bullet_+) = j \xrightarrow{e} Z(\bullet_+) \xrightarrow{Z(I_n)} Z(\bullet_+) \xrightarrow{r} j = Z^{\mathrm{new}}(\bullet_+) \right] .$$

Of course, $Z^{\mathrm{new}}(I)$ is just the identity on j, as it should be. (Check this.) In particular, it is independent of n. ?

This is an example of a state-sum construction of a TFT. But the 1d case is too simple to appreciate what is going on. It is better to discuss the 2d case first and then return to the 1d case to see that this is indeed the same idea.

For info: Given a lattice model, one can try and obtain a continuum field theory by passing to the "continuum limit" by trying to approximate the continuum manifold by finer and finer lattices. Intuitively, this exhibits the long range behaviour of a lattice system, e.g. correlations over a large number of lattice spacings, and depends less on the microscopic details. Different lattice models can have the same continuum limit (if it exists at all). The field theory obtained from the continuum limit is for this reason also called the "universality class". For state sum TFTs as in the simple 1d example above, one can take the continuum limit *exactly*, and trivially, as by design the lattice model is actually independent of the lattice size.

Bibliographic notes for Section 3

The description of 1d TFTs via generators and relations is a first example in many places, see for example [Te, Sec. 4] or [CR2, Sec. 3.1].

The particular approach to stratified bordisms via maps into a poset is taken from [Lu2, App. A.5] and [AFT, Sec. 2.1]. A related description of stratified bordisms is given in [CMS, Sec. 2]. A more detailed discussion of stratified n-dimensional PL-bordisms and of the corresponding TFTs will be given in [CM].

Some pointers for state sum models are included in the bibliographic notes for Section 4.

4 Dimension two

4.1 Frobenius algebras

Definition 4.1. Let C be a monoidal category.

1. A *monoid*, or a *(unital, associative) algebra*, in C is an object $A \in C$, together with morphisms $\mu : A \otimes A \to A$ and $\eta : \mathbb{1} \to A$ such that the following diagrams commute:

   - associativity:

$$\begin{tikzcd} A \otimes (A \otimes A) \arrow[r, "1_A \otimes \mu"] \arrow[dd, "\alpha_{A,A,A}"] & A \otimes A \arrow[dr, "\mu"] & \\ & & A \\ (A \otimes A) \otimes A \arrow[r, "\mu \otimes 1_A"] & A \otimes A \arrow[ur, "\mu"'] & \end{tikzcd}$$

   - unitality:

$$\begin{tikzcd} & \mathbb{1} \otimes A \arrow[r, "\eta \otimes 1_A"] & A \otimes A \arrow[dr, "\mu"] & \\ A \arrow[ur, "\lambda_A^{-1}"] \arrow[rrr, "1_A"] \arrow[dr, "\rho_A^{-1}"'] & & & A \\ & A \otimes \mathbb{1} \arrow[r, "1_A \otimes \eta"] & A \otimes A \arrow[ur, "\mu"'] & \end{tikzcd}$$

2. A *monoid, or algebra, morphism* is a morphism $f : A \to A'$ in C such that
$$\mu' \circ (f \otimes f) = f \circ \mu \quad , \quad f \circ \eta = \eta' \ .$$

3. A *comonoid*, or *coalgebra* in C is a monoid in C^{opp}. Explicitly, it an object $B \in C$ with morphism $\Delta : B \to B \otimes B$ and $\varepsilon : B \to \mathbb{1}$ in C such that the above diagrams hold with reversed arrows. (Write these out.) ?

4. A *comonoid, or coalgebra, morphism* is ... (Complete the definition.) ?

Example 4.2. A monoid in Set is a monoid. A monoid in the category $\mathbb{Z}$-mod is a ring. A monoid in Vec_k is an (associative, unital) k-algebra. (Check all these.) ?

The last example is the reason why in k-linear categories, and also in general, one often uses the term "algebra" instead of monoid. We will use "(co)algebra" below, even when we are in a non-k-linear category. ?

(Try to impress your computer scientist friends by saying that a monad is simply a monoid in the category of endofunctors. But first make sure this is correct.)

Example 4.3. Take $C = \mathsf{Vec}_G^\omega$ for $\omega \in Z^3(G, k^\times)$ normalised. Recall that k_g, $g \in G$, denotes the 1-dimensional vector space k, concentrated in degree g. Consider $A = \bigoplus_{g \in G} k_g$, and write $b_g = 1 \in k_g$ for the corresponding basis. Define

$$\mu : A \otimes A \to A \ , \quad b_g \otimes b_h \mapsto \varphi(g,h) b_{gh} \ , \quad \eta : k_e \to A \ , \quad 1 \mapsto b_e \ ,$$

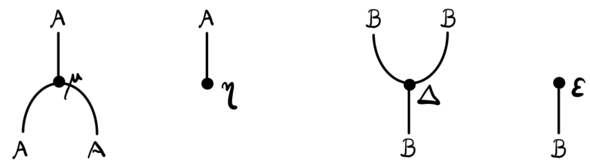
for some $\varphi \in C^2(G, k^\times)$ normalised. (Why are these morphisms in Vec_G^ω?) ?
Because ω and φ are normalised, the unitality conditions hold. The associativity condition, applied to $b_f \otimes (b_g \otimes b_h)$, reads

$$\varphi(f, gh)\, \varphi(g,h) \ = \ \varphi(fg, h)\, \varphi(f,g)\, \omega(f,g,h) \ .$$

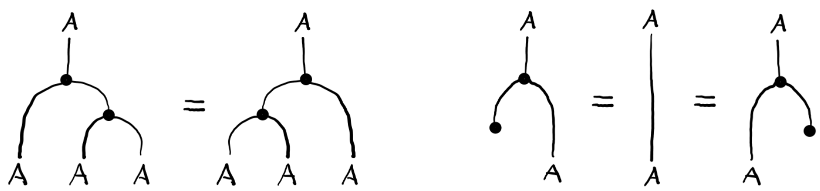
In other words, $\omega = d\varphi$. We can conclude that

1. A admits an associative product of the above form if and only if $[\omega]$ is trivial in $H^3(G, k^\times)$, and
2. in that case there are $H^2(G, k^\times)$ many isomorphism classes of algebras with such a product. (Actually we did not show that. Why is it true?) ?

■ It will be helpful to have graphical notation for the morphisms in an algebra A and a coalgebra B:



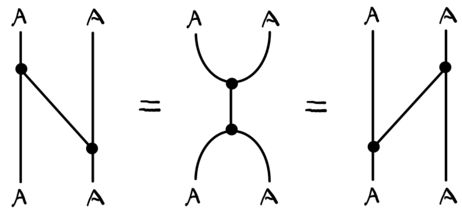
When using graphical calculus, we implicitly work in a strict monoidal category. The associativity and unitality conditions become



(Write out the coassociativity and counitality.) ?

Definition 4.4. Let C be a monoidal category.

1. A *Frobenius algebra* in C is a tuple $(A, \mu, \Delta, \eta, \varepsilon)$ so that (A, μ, η) is an algebra, (A, Δ, ε) is a coalgebra, and the *Frobenius relations* hold:



(These pictures make sense in the strictification of C. Write out the corresponding commuting diagrams in C with all coherences included.) ?

2. A *morphism of Frobenius algebras* is a morphism $f : A \to B$ in C which is a morphism of the underlying algebra and coalgebra of A and B.

Lemma 4.5. A morphism of Frobenius algebras is invertible.

The proof of this lemma is part of the following exercise:

Exercise 4.6. Let A be a Frobenius algebra in a monoidal category C. Show that A is its own left and right dual. Conclude that a morphism $f : A \to B$ of Frobenius algebras is necessarily invertible. *Hint:* Try $\varepsilon \circ \mu$ and $\Delta \circ \eta$ for left/right (co)evaluation morphisms. For invertibility proceed analogous to the proof of Lemma 2.52. E

■ By the above exercise, if $(A, \mu, \Delta, \eta, \varepsilon)$ is a Frobenius algebra in C, then the underlying object A is dualisable, $A \in C^{\mathrm{fd}}$. Hence, for example, every Frobenius algebra in Vec_k is finite-dimensional.

■ An alternative definition of a Frobenius algebra is as follows: A Frobenius algebra is an algebra (A, μ, η) together with a morphism $\varepsilon : A \to \mathbb{1}$, such that $\varepsilon \circ \mu : A \otimes A \to \mathbb{1}$ is a non-degenerate pairing. The latter property means that there is a copairing $\gamma : \mathbb{1} \to A \otimes A$ satisfying the two zig-zag identities. For C strict monoidal, these read

$$\left[A \xrightarrow{\gamma\otimes 1_A} AAA \xrightarrow{1_A\otimes(\varepsilon\circ\mu)} A\right] = 1_A = \left[A \xrightarrow{1_A\otimes\gamma} AAA \xrightarrow{(\varepsilon\circ\mu)\otimes 1_A} A\right]$$

Let us call this notion a Frobenius$_2$ algebra for now. We would like to show that a Frobenius$_2$ algebra is the same as a Frobenius algebra.

One direction we did already do. Namely, let $(A, \mu, \Delta, \eta, \varepsilon)$ be a Frobenius algebra as in Definition 4.4. In Exercise 4.6 we saw that $(A, \mu, \eta, \varepsilon)$ is a Frobenius$_2$ algebra. The converse direction is the next exercise:

Exercise 4.7. Let now $(A, \mu, \eta, \varepsilon)$ be a Frobenius$_2$ algebra. We compute in the strictification of C. (E)

1. Define

$$D_1 = \Big[A \xrightarrow{\gamma \otimes 1_A} AAA \xrightarrow{1_A \otimes \mu} AA \Big] \ , \quad D_2 = \Big[A \xrightarrow{1_A \otimes \gamma} AAA \xrightarrow{\mu \otimes 1_A} AA \Big] \ .$$

   Show that $D_1 = D_2$. *Hint:* Check that $((\varepsilon \circ \mu) \otimes 1_A) \circ (1_A \otimes D_i) = \mu$ for $i = 1, 2$. It helps to do this in graphical calculus.

2. Set $\Delta = D_1 = D_2$. Show that Δ is coassociative, counital, and satisfies the Frobenius property. *Hint:* The key to these computations is to replace Δ by D_1 or D_2 cleverly.

Thus, together with Δ as above, A is a Frobenius algebra in the sense of Definition 4.4.

Example 4.8. 1. Let k be a field and $\delta \in k^\times$. Then $k \in \mathsf{Vec}_k$ is already an algebra. We turn it into a Frobenius algebra k_δ by defining a coproduct and counit by

$$\Delta : k \mapsto k \otimes k \ , \quad 1 \mapsto \delta^{-1} 1 \otimes 1 \quad , \quad \varepsilon : k \mapsto k \ , \quad 1 \mapsto \delta \, 1 \ .$$

   Then $k_\gamma \cong k_\delta$ as Frobenius algebras iff $\gamma = \delta$, and every one-dimensional Frobenius algebra in Vec_k is isomorphic to one of the k_δ. (Why?) (?)

2. Consider $\mathrm{Mat}_n(K)$, the algebra of $n \times n$-matrices with entries in a commutative ring K. Take

$$\varepsilon : \mathrm{Mat}_n(K) \to K \ , \quad M \mapsto \mathrm{tr}(M) \ .$$

   The trace pairing $(M, N) \mapsto \mathrm{tr}(MN)$ is non-degenerate, and so this defines a Frobenius algebra via Exercise 4.7. Let E_{ij} be the standard basis of $\mathrm{Mat}_n(K)$, that is, the matrix with entries $(E_{ij})_{ab} = \delta_{i,a}\delta_{j,b}$. The coproduct is

$$\Delta(M) = \sum_{i,j=1}^{n} (M E_{ij}) \otimes_K E_{ji} = \sum_{i,j=1}^{n} E_{ij} \otimes_K (E_{ji} M) \ .$$

   (Why is the trace pairing non-degenerate? Check explicitly that the two expressions for the coproduct agree by expanding also M in the E_{ij}-basis. Check that Δ is indeed the coproduct obtained via Exercise 4.7. *Hint:* First check that the copairing is $\gamma = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji}$.) (?)

3. Let G be a finite group and consider the group algebra $k[G]$ from Exercise 2.38 (2). Then $\varepsilon : k[G] \to k$, $b_g \mapsto \delta_{g,e}$ induces a non-degenerate pairing. (What is the coproduct? What, if anything, changes if we twist by a normalised 2-cocycle $\varphi \in Z^2(G, k^\times)$ to get $k[G]^\varphi$? *Hint:* Try the copairing $\gamma = \sum_g b_g \otimes_k b_{g^{-1}}$.) ?

4. Let $k[X]$ be the polynomial ring and $\langle X^d \rangle = X^d k[X]$ be the ideal generated by X^d. Consider the quotient $F = k[X]/\langle X^d \rangle$. Define $\varepsilon(X^m) = \delta_{m,d-1}$. This defines a non-degenerate pairing, and so F is a Frobenius algebra. (What is the coproduct in this case? What fails if we take $\varepsilon(X^m) = \delta_{m,0}$? Can you think of other choices for ε?) ?

Definition 4.9. Let C be a braided monoidal category. An algebra A is *commutative* (resp. a coalgebra B *cocommutative*), if $\mu \circ \beta_{A,A} = \mu$ (resp. $\beta_{B,B} \circ \Delta = \Delta$).

Example 4.10. Let V be a finite-dimensional k-vector space and $F = \Lambda(V)$ the exterior algebra. This is a $\mathbb{Z}$-graded, non-commutative k-algebra, concentrated in degrees $0, \dots, \dim(V)$. Turn F into a super-vector space by declaring even graded components even, and odd graded components odd.

Equivalently, take V a purely odd vector space in SVec and consider the symmetric algebra $F = S(V)$ in SVec. (Why is that the same?) ?

If $\dim(V)$ is even, then F is a commutative Frobenius algebra in SVec_k by taking $\varepsilon : F \to k$ to be non-zero only in degree n, the one-dimensional space of top forms. (What fails if V is odd-dimensional?) ?

Exercise 4.11. Show that a Frobenius algebra is commutative if and only if it is cocommutative. E

(30.4.26)

Another thing we can do if the category we are working in is braided, apart from defining commutativity, is taking products of algebras and coalgebras.

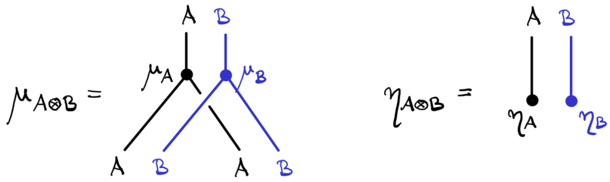
Lemma 4.12. Let C be braided monoidal.

1. Let A, B be algebras in C. Then $A \otimes B$ is an algebra in C via (omitting "$\otimes$" between objects)

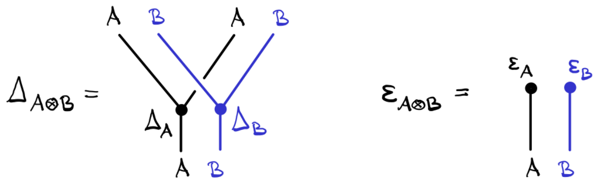
$$\mu_{A\otimes B} = \Big[(AB)(AB) \xrightarrow{\sim} A((BA)B) \xrightarrow{1_A \otimes (\beta_{B,A} \otimes 1_B)} A((AB)B) \xrightarrow{\sim} (AA)(BB) \xrightarrow{\mu_A \otimes \mu_B} AB\Big]\,,$$

$$\eta_{A\otimes B} = \Big[\mathbb{1} \xrightarrow{\sim} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\eta_A \otimes \eta_B} AB\Big]\,.$$

In graphical notation and for C strict, this reads

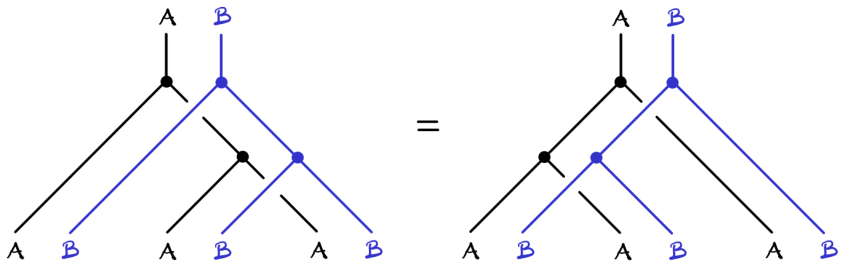


2. If A, B are coalgebras in C, then $A \otimes B$ is a coalgebra via (in graphical notation for C strict)

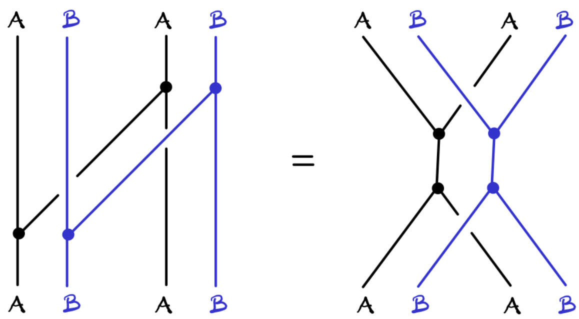


3. If A, B are Frobenius algebras, then $A \otimes B$ with (co)algebra structure as above is again a Frobenius algebra.

Proof. We check two of the required identities in graphical calculus, the rest is done analogously. For this we assume C to be strict. In part 1, associativity of $\mu_{A\otimes B}$ amounts to the identity



which follows from the properties of the braiding. (Try to insert the intermediate steps and identify which property of the braiding is being used.) ?
One of the Frobenius relations in part 3 can analogously be verified from the braiding,



□

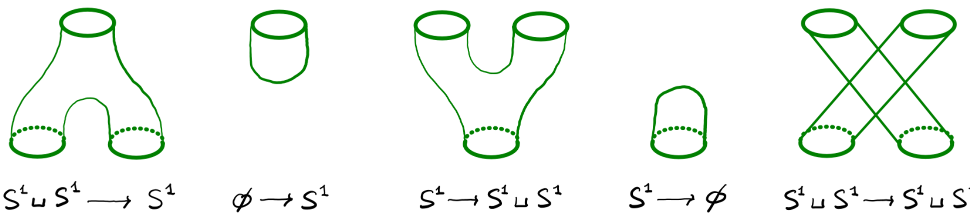
■ Note that the definition of the product on $A \otimes B$ involves the braiding $\beta_{B,A}$, while the coproduct uses the inverse braiding $\beta^{-1}_{B,A}$. From an algebraic point of view, this may seem arbitrary, but from the graphical calculus point of view, it amounts to the natural prescription "2nd factor on top".

As far as only the (co)algebra structure in parts 1 and 2 is concerned, in each case one could choose the braiding or its inverse, this is a convention. However, to get a Frobenius algebra in part 3, the two choices must be correlated as either "2nd factor on top" or "1st factor on top". Here we went with the first option. (Check that using $\beta_{A,B}$ for the coproduct (instead of $\beta^{-1}_{B,A}$) breaks the Frobenius relations on $A \otimes B$.) ?

4.2 Generators and relations

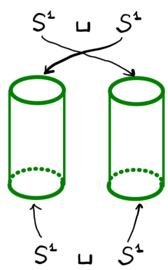
We again import two results from topology. The first is a list of generators for Bord_2. The second result concerns the relations among these generators and will be used in the proof of the theorem below. Both can be shown via Morse theory or via a normal form. See [Ko, Sec. 1.4] for more details on the latter.

Proposition 4.13. The category Bord_2 is tensor generated by $C_0 = \{S^1\}$ and



(pants) (cup) (copants) (cap) (crossing)

■ The crossing bordism in the last picture is not drawn as an over- or under-crossing as these 2-bordisms are not embedded in some $\mathbb{R}^3$. A better way to represent the crossing may be to indicate the different ways $S^1 \sqcup S^1$ is glued to the boundary of $(S^1 \sqcup S^1) \times I$:

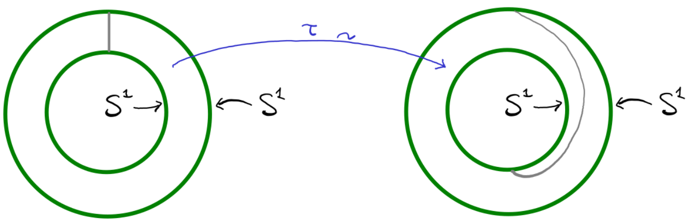


■ For a braided monoidal category C, denote by $\mathsf{ComFrob}(C)$ the category with objects commutative Frobenius algebras in C and morphisms of Frobenius algebras between these objects. By Lemma 4.5, $\mathsf{ComFrob}(C)$ is a groupoid.

Theorem 4.14. Let S be a symmetric monoidal category. Then $\mathsf{TFT}_2(S) \cong \mathsf{ComFrob}(S)$ via $Z \mapsto (A, \mu, \Delta, \eta, \varepsilon)$, where $A = Z(S^1)$ and

$$\begin{aligned}
\mu &= \Big[A \otimes A = Z(S^1) \otimes Z(S^1) \xrightarrow{\sim} Z(S^1 \sqcup S^1) \xrightarrow{Z(\text{pants})} Z(S^1) = A\Big] , \\
\eta &= \Big[\mathbb{1} \xrightarrow{\sim} Z(\emptyset) \xrightarrow{Z(\text{cup})} Z(S^1) = A\Big] , \\
\Delta &= \Big[A = Z(S^1) \xrightarrow{Z(\text{copants})} Z(S^1 \sqcup S^1) \xrightarrow{\sim} Z(S^1) \otimes Z(S^1) = A \otimes A\Big] , \\
\varepsilon &= \Big[A = Z(S^1) \xrightarrow{Z(\text{cap})} Z(\emptyset) \xrightarrow{\sim} \mathbb{1}\Big] .
\end{aligned}$$

Proof (sketch): A as above is indeed a commutative Frobenius algebra: Let us do commutativity and one of the Frobenius relations as representative examples. We work with the strictifications of Bord_2, S and Z to avoid writing all coherences. For commutativity, we use the following isomorphism:



The left picture is the identity bordism over S^1, and the gray line is included to illustrate the isomorphism τ, which rotates concentric circles by angles increasing from 0 to π. Hence, on the right hand side the inner circle, with its parametrisation, got rotated by π relative to the outer one. The two bordisms are isomorphic via τ, and hence equal in the bordism category.

Commutativity follows from constructing an isomorphism between the pair of pants and the pair of pants composed with crossing, see Figure 4.1. The isomorphism in the figure ϕ is given by τ on the outer annulus, and by the constant rotation by π on the inner disc. The isomorphism ψ is given by the identity outside the two inner dashed annuli, and on each of the two annuli it is τ^{-1}. As ψ and ϕ are isomorphisms, the value of Z^{str} stays the same. The last bordism is equal to the pair of pants composed with the crossing bordism (the last generator shown in Proposition 4.13). Note that this is not the case for the second bordism as the parametrisation of the two inner boundary circles is rotated by π. (Technically, one needs to use a PL-version of all these isomorphisms.)

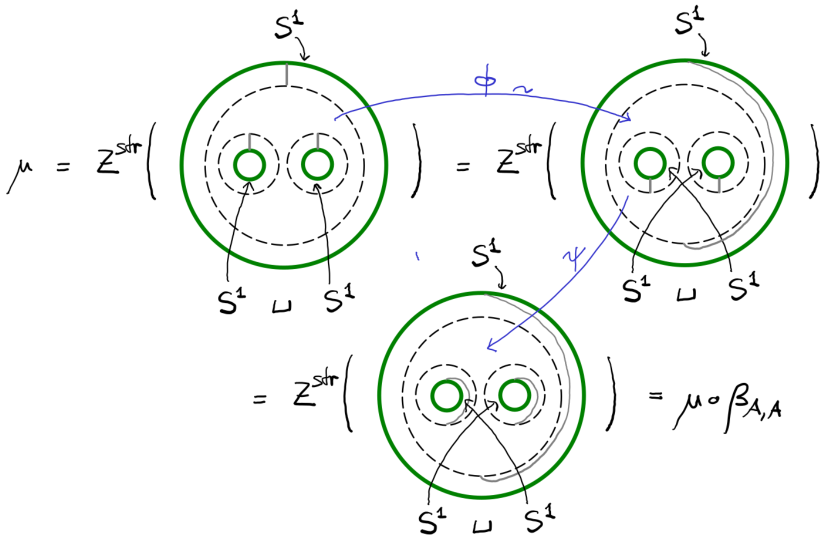
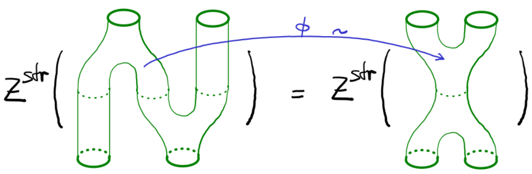


Figure 4.1: Demonstration of $\mu = \mu \circ \beta_{A,A}$

For one of the Frobenius relations one proceeds analogously. Again the middle equality is an isomorphism of bordisms.

$$(\mu \otimes 1_A) \circ (1_A \otimes \Delta) \quad =$$

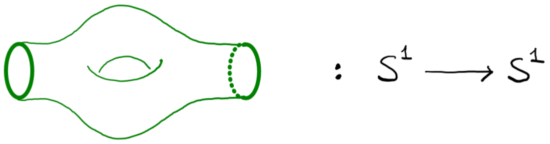


$$= \quad \Delta \circ \mu$$

For a natural transformation $\phi : Y \Rightarrow Z$, the restriction $\phi_{S^1} : Y(S^1) \to Z(S^1)$ does indeed define a morphism of Frobenius algebras. Conversely, every morphism of Frobenius algebras extends to a natural transformation via Lemma 3.4.

Thus the functor in the statement of the theorem is fully faithful. It remains to see that it is essentially surjective. In fact, the functor is even surjective, that is, every commutative Frobenius algebra can occur as $Z(S^1)$, not just up to isomorphism. This part of the argument is why the proof is a sketch, as we will import this statement from topology: The relations among the generators are precisely that they define a commutative Frobenius algebra in Bord_2, see [Ko, Sec. 1.4] for details. □

■ Let us compute the value of a TFT Z on a closed genus-g surface Σ_g for some examples of commutative Frobenius algebras. Let H be the the 2-torus with one incoming and one outgoing boundary component,



We can decompose Σ_g as H^g composed with cup and cap

Let $A = (A, \mu, \eta, \Delta, \varepsilon)$ be the commutative Frobenius algebra in S obtained from Theorem 4.14. We have (here and for the rest of this section we assume that $Z^0 = 1_{\mathbb{1}}$, and so $Z(\emptyset) = \mathbb{1}$, to simplify notation)

$$Z(\Sigma_g) = \varepsilon \circ h^g \circ \eta \qquad \text{where} \quad h = Z(H) = \mu \circ \Delta : A \to A \ .$$

Example 4.15. We keep the numbering from Example 4.8.

1. For k_δ we have $h = \delta^{-1}$, and so
$$Z(\Sigma_g) = \varepsilon \circ h^g \circ \eta = \delta^{1-g} = \delta^{\chi(\Sigma_g)/2} \ ,$$
where $\chi(\Sigma_g) = 2 - 2g$ is the Euler characteristic of Σ_g.

2. $\mathrm{Mat}_n(K)$ is not commutative.

3. $k[G]$ is commutative for G abelian, and one finds $h = |G| \, \mathrm{id}$, and so $Z(\Sigma_g) = |G|^g$. Note that $Z(\Sigma_g) \in k$, and so if $\mathrm{char}(k)$ divides $|G|$, then $Z(S^2) = 1$ and $Z(\Sigma_g) = 0$ for $g \geq 1$.

4. For $a \in k[X]/\langle X^d \rangle$ we have $h(a) = d\, X^{d-1} a$ (see exercise below). Assuming $d > 1$, we have $Z(\Sigma_g) = d\, \delta_{g,1}$.

■ A TFT $Z \in \mathsf{TFT}_2(\mathsf{Vec}_{\mathbb{C}})$ defines a sequence of numbers $x_g \in \mathbb{C}$, $g = 0, 1, 2, \dots$ by setting $x_g = Z(\Sigma_g)$. One can conversely ask if such a sequence determines a 2d TFT up to equivalence, and if any such sequence can occur. The answer to both questions is negative, but for the second question one can give a necessary condition. This is the content of the next exercise.

Exercise 4.16. 1. Check that for $a \in k[X]/\langle X^d \rangle$ we have $h(a) = d\, X^{d-1} a$. E
To do so, first compute the copairing γ for the map ε given in Example 4.8 (4). Then compute the coproduct, and then $h = \mu \circ \Delta$.

2. Give two non-equivalent TFTs $Y, Z \in \mathsf{TFT}_2(\mathsf{Vec}_{\mathbb{C}})$ such that $Y(\Sigma_g) = Z(\Sigma_g)$ for all g. *Hint:* Write $F_m = \mathbb{C}[X]/\langle X^m \rangle$. Try $F_m \oplus F_n$.

3. Let $x_g \in \mathbb{C}$, $g = 0, 1, 2, \dots$ be a sequence of complex numbers. Show: If there is a TFT $Z \in \mathsf{TFT}_2(\mathsf{Vec}_{\mathbb{C}})$ satisfying $x_g = Z(\Sigma_g)$, then there is $r > 0$ and $\lambda_1, \dots, \lambda_r \in \mathbb{C}$ such that for all $g \geq 2$,

$$x_g = \sum_{k=1}^{r} \lambda_k^{g-1} .$$

Hint: Show that for $g \geq 2$ one has $Z(\Sigma_g) = \mathrm{tr}(h^{g-1})$. Consider the Jordan normal form of h.

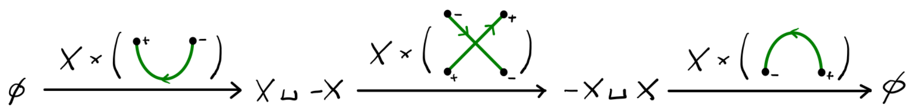
■ The above exercise gave conditions for x_g, $g \geq 2$. While the argument used in the proof does not apply to $g = 1$, the formula would predict $x_1 = r$ for some $r \in \mathbb{Z}_{>0}$. In fact, one always has $x_1 = \dim_{\mathbb{C}}(Z(S^1))$. This is a consequence of the following general property of n-dimensional TFTs.

Proposition 4.17. Let $Z \in \mathsf{TFT}_n(S)$. Then for all $X \in \mathsf{Bord}_n$ we have

$$Z(X \times S^1) = \dim(Z(X)) .$$

as elements of $\mathrm{End}(\mathbb{1})$.

Proof. We assume S to be strict. Write $X \times S^1$ as a composition of three bordisms as follows:



The first and last morphism are the left (co)evaluation in Bord_n (Proposition 2.48), and so after applying Z, these become left (co)evaluation in S (Proposition 2.24). Together with the definition of the dimension from above Proposition 2.33, we get

$$Z(X \times S^1) = \Big[\mathbb{1} \xrightarrow{\mathrm{coev}_{Z(X)}} Z(X) \otimes Z(-X) \xrightarrow{\beta_{Z(X),Z(-X)}} Z(-X) \otimes Z(X) \xrightarrow{\mathrm{ev}_{Z(X)}} \mathbb{1} \Big] = \dim(Z(X)) .$$

□

4.3 Stacking and invertible TFTs

Let S be a symmetric monoidal category. Given $Y, Z \in \mathsf{TFT}_n(S)$, we can define a new TFT $Y \otimes Z \in \mathsf{TFT}_n(S)$ by "stacking", turning TFT_n itself into a symmetric monoidal category. (The term stacking comes from the physical interpretation of having a two-layer system, with one layer given by Y and one by Z.)

The following exercise gives a step-by-step description of $Y \otimes Z$.

Exercise 4.18. 1. Let us look at an example one categorical level lower first. If M is a monoid (in Set, i.e. a set with a unit and an associative product), the for any other set X, the set of maps $\mathrm{Map}(X, M)$ is again a monoid: For $f, g : X \to M$ set $(f \cdot g)(x) = f(x) \cdot g(x)$. Ⓔ

If X is also a monoid, and $f, g : X \to M$ are monoid morphisms (preserve unit and multiplication), give an example, where $f \cdot g$ is not a monoid map. Show that this problem goes away if we demand M to be commutative.

2. Let C be a category and M a monoidal category. Given two functors $F, G : C \to M$, define a new functor $F \otimes G : C \to M$.

3. Suppose now that C is monoidal and S is symmetric monoidal. Let $F, G : C \to S$ be monoidal functors. Show that $F \otimes G : C \to S$ as above can be equipped with the structure of a monoidal functor. *Hint:* For $(F \otimes G)^2$ try

$$\begin{aligned}
&(F \otimes G)(A) \otimes (F \otimes G)(B) = (F(A) \otimes G(A)) \otimes (F(B) \otimes G(B)) \\
&\xrightarrow{\sim} (F(A) \otimes F(B)) \otimes (G(A) \otimes G(B)) \\
&\xrightarrow{\sim} F(A \otimes B) \otimes G(A \otimes B) = (F \otimes G)(A \otimes B) \ .
\end{aligned}$$

What is $(F \otimes G)^0$?

4. Show that if C, S are symmetric monoidal, and $F, G : C \to S$ are symmetric monoidal, so is $F \otimes G$. Show that in this case $F \otimes G \cong G \otimes F$ as monoidal functors.

Definition 4.19. 1. The *trivial TFT* $I \in \mathsf{TFT}_n(S)$ is TFT sending all objects in Bord_n to the tensor unit, and all bordisms to the identity morphism: $I : \mathsf{Bord}_n \to S$, $I(X) = \mathbb{1}$, $I(X \xrightarrow{M} Y) = [\mathbb{1} \xrightarrow{1_{\mathbb{1}}} \mathbb{1}]$. (Why is this a TFT?) ?

2. A TFT $Z \in \mathsf{TFT}_n(S)$ is *invertible* if there is $Y \in \mathsf{TFT}_n(S)$ such that $Z \otimes Y \cong I$, or, equivalently, $Y \otimes Z \cong I$ in $\mathsf{TFT}_n(S)$.

(5.5.26)

■ An object $U \in S$ is called *tensor-invertible* (or just *invertible*) if there is an object $V \in S$ such that $U \otimes V \cong \mathbb{1}$ (which implies $V \otimes U \cong \mathbb{1}$ as S is symmetric). Note that this definition makes sense in any monoidal category.

Lemma 4.20. Let $Z \in \mathsf{TFT}_n(S)$ be invertible. Then $Z(X)$ is tensor-invertible in S for every $X \in \mathsf{Bord}_n$.

Proof. Let $Y \in \mathsf{TFT}_n(S)$ satisfy $Z \otimes Y \cong I$. Let $\eta : Z \otimes Y \Rightarrow I$ be the corresponding natural monoidal isomorphism. Then, for each $X \in \mathsf{Bord}_n$,

$$Z(X) \otimes Y(X) = (Z \otimes Y)(X) \xrightarrow{\eta_X} I(X) = \mathbb{1} \ .$$

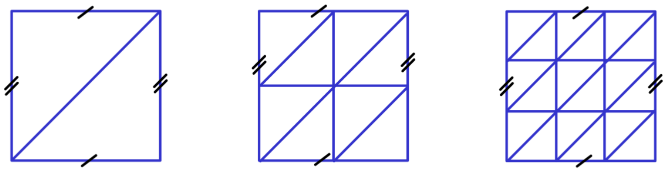
□

For example if $S = \mathsf{Vec}_k$, an invertible TFT necessarily has one-dimensional state spaces $Z(X)$ for all objects $X \in \mathsf{Bord}_n$. The TFT obtained from the commutative Frobenius algebra k_δ in Example 4.15 (1) is invertible with inverse $k_{1/\delta}$.

■ The above example generalises to all all dimensions, giving so-called *Euler topological field theories* which on closed manifolds M compute the Euler characteristic of M. We first need:

Definition 4.21. Let M be an n-dimensional compact (PL-)manifold, possibly with non-empty boundary. A *triangulation* of M is a decomposition of M into n-simplices such that the intersection $\sigma_1 \cap \sigma_2$ of any two (sub)simplices σ_1, σ_2 is a face of both of them.

For example, of the following three decompositions of a 2-torus into 2-simplices, only the last one is a triangulation. (What fails for the other two?) ?



■ Recall that the *Euler characteristic* $\chi(M)$ with M as in the above definition is the alternating sum

$$\chi(M) = \sum_{k=0}^{n} (-1)^k N_k \ ,$$

where N_k is the number of k-simplices in the triangulation. In the example of the torus above, being careful not to overcount at the edges which are identified, we get

$$\chi(T^2) = N_0 - N_1 + N_2 = 9 - 27 + 18 = 0 \ .$$

Importantly, $\chi(M)$ does not depend on the choice of triangulation. We collect some more of its properties:

$$\chi(M \times I) = \chi(M) \quad , \quad \chi(M \times S^1) = 0 \quad , \quad \chi(D^n) = 1$$

(and so indeed for $T^2 = S^1 \times S^1$ we must get zero). Furthermore, for M, N be compact n-manifolds and Y a common $n-1$ dimensional submanifold on ∂M and ∂N, we have

$$\chi(M \sqcup_Y N) = \chi(M) + \chi(N) - \chi(Y) \ .$$

This allows one to compute the Euler characteristics of spheres to be

$$\chi(S^n) = 1 + (-1)^n \ .$$

(Check this by induction.) Note that this is zero if n is odd. This is actually a general fact which follows from Poincaré duality [Ha, Cor. 3.37] ?

$$\chi(M) = 0 \quad \text{for a closed odd-dimensional manifold } M \ .$$

■ The behaviour of χ under gluing motivates the following modified definition of the Euler character of a bordism $M : X \to Y$ in Bord_n:

$$\chi_b(M) = \chi(M) - \chi(X) \ .$$

We could have equally well chosen $\chi(M) - \chi(Y)$ or any convex combination of $\chi(X)$ and $\chi(Y)$, but let us stick with this convention. Given bordisms $M : X \to Y$ and $N : Y \to Z$ we get (Check this.) ?

$$\chi_b(N \circ M) = \chi_b(N) + \chi_b(M) \ .$$

Proposition 4.22. Let S be a symmetric monoidal category, and let $\delta \in \mathrm{End}(\mathbb{1})$ be invertible. Then

$$E_\delta : \mathsf{Bord}_n \to S \quad , \quad X \mapsto \mathbb{1} \ , \quad (X \xrightarrow{M} Y) \mapsto \delta^{\chi_b(M)} \ ,$$

defines an n-dimensional invertible TFT, called *Euler topological field theory*. It satisfies $E_\delta \otimes E_\varepsilon \cong E_{\delta\varepsilon}$.

Proof. We first check that E_δ is a symmetric monoidal functor, and then $E_\delta \otimes E_\varepsilon \cong E_{\delta\varepsilon}$. Since $E_1 = I$, this implies $E_\delta \otimes E_{\delta^{-1}} \cong I$, and so E_δ is indeed invertible.

E_δ is a functor: For the unit morphism $X \times I$ we find $\chi_b(X \times I) = \chi(X \times I) - \chi(X) = 0$. For compatibility with composition, take $M : X \to Y$ and $N : Y \to Z$

$$E_\delta(N) \circ E_\delta(M) = \delta^{\chi_b(N)}\,\delta^{\chi_b(M)} = \delta^{\chi_b(N)+\chi_b(M)} = \delta^{\chi_b(N\circ M)} = E_\delta(N \circ M)\ .$$

E_δ is symmetric monoidal: The monoidal structure on E_δ is simply

$$\begin{aligned}(E_\delta)^2_{X,Y} &= \Big[E_\delta(X) \otimes E_\delta(Y) = \mathbb{1} \otimes \mathbb{1} \xrightarrow{\sim} \mathbb{1} = E_\delta(X \sqcup Y)\Big]\ ,\\ (E_\delta)^0 &= \Big[\mathbb{1} \xrightarrow{1_\mathbb{1}} \mathbb{1} = E_\delta(\emptyset)\Big]\ .\end{aligned}$$

That $(E_\delta)^2_{X,Y}$ is natural in X and Y follows from additivity of χ_b under disjoint union. The commuting diagrams for a monoidal functor (Definition 2.23) and for compatibility with the braiding (Definition 2.32) are just coherence diagrams for $\mathbb{1} \in S$ and hold automatically.

Multiplicativity $E_\delta \otimes E_\varepsilon \cong E_{\delta\varepsilon}$: We need to give a monoidal natural isomorphism $\eta_X : E_\delta \otimes E_\varepsilon \Rightarrow E_{\delta\varepsilon}$. We take

$$\eta_X = \Big[E_\delta(X) \otimes E_\varepsilon(X) = \mathbb{1} \otimes \mathbb{1} \xrightarrow{\sim} \mathbb{1} = E_{\delta\varepsilon}(X)\Big]\ .$$

The naturality square for a bordism $X \xrightarrow{M} Y$ reads

$$\begin{tikzcd} E_\delta(X) \otimes E_\varepsilon(X) \arrow[r, "E_\delta(M)\otimes E_\varepsilon(M)"] \arrow[d, "\eta_X"'] & E_\delta(Y) \otimes E_\varepsilon(Y) \arrow[d, "\eta_Y"] \\ E_{\delta\varepsilon}(X) \arrow[r, "E_{\delta\varepsilon}(M)"] & E_{\delta\varepsilon}(Y) \end{tikzcd}$$

and simply becomes the identity $\delta^{\chi_b(M)}\varepsilon^{\chi_b(M)} = (\delta\varepsilon)^{\chi_b(M)}$ in $\mathrm{End}(\mathbb{1})$. Monoidality of η_X amounts to checking the square in Definition 2.26. This is slightly lengthy to write down in detail because of the reshuffling of factors in the monoidal structure of $E_\delta \otimes E_\varepsilon$ in Exercise 4.18 (3). But in the end both sides are just compositions of coherence isomorphisms for $\mathbb{1} \in S$, and these automatically agree (use Exercise 2.30 for $\beta_{\mathbb{1},\mathbb{1}}$). □

For info: Classifying invertible topological field theories relates to interesting questions in topology, in particular to cobordisms rings. The degree d component of the oriented cobordism ring Ω^{SO}_* consists of all d-manifolds X, up to the equivalence relation $X \sim Y$ that $X \sqcup -Y$ is the boundary of an $(d+1)$-dimensional oriented manifold. The first few degrees are:

d	0	1	2	3	4	5	6
Ω^{SO}_d	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0

The generator of $\mathbb{Z}$ in Ω_0^{SO} is the positively oriented point – only pairs $\bullet_+ \sqcup \bullet_-$ are cobordant to the empty set. Any oriented circle is the boundary of a disc, and so indeed $\Omega_1^{SO} = 0$. Similarly, a genus-g-surface is the boundary of the corresponding handle body, giving $\Omega_2^{SO} = 0$. The other entries are more non-trivial.

There are also other cobordism rings, such as Ω_*^O for unoriented manifolds, or Ω_*^{Spin} for manifolds with spin structure. For more on cobordism groups see [MSt, §17], and for more on the relation between cobordism groups and TFTs, see [Sz, Sec. 2] and [RSch, Thm. 3.15, Ex. 3.18]. The latter reference describes the specific cobordism groups which classify invertible topological field theories.

4.4 Trivial n-dimensional TFT with line defects

Let S be a rigid symmetric monoidal category, that is, every object is dualisable:

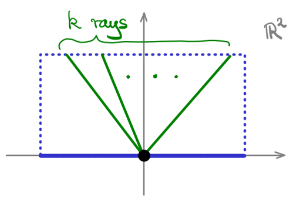
$$S = S^{\mathrm{fd}} .$$

We would like to define an n-dimensional TFT with line defects (1-strata) coloured by objects of S and point defects (0-strata) coloured by morphisms of S. The surrounding n-dimensional theory should be trivial in the sense that the value of the TFT does not depend on the n-dimensional bordism, but just the 1-strata.

- We take $D(S) = D_0 \sqcup D_1 \sqcup D_n$, with $D_0 = \mathrm{mor}(S)$, the class of morphisms in S, $D_1 = \mathrm{obj}(S)$, the class of objects in S, and $D_n = \{*\}$, the 1-element set. The map $\pi : D \to \{0, 1, \dots, n\}$ takes elements of D_k to k. The partial order is simply $D_0 < D_1 < D_n$.

- We would like the morphism $f \in D_0$ labelling a 0-stratum to dictate how the neighbouring 1-strata are labelled by objects of S.

Write H_k for the 2-dimensional open unit disc intersected with the upper half plane (or rather the PL-version, a open unit square), together with k rays from 0 to the upper boundary,

$H_k =$ 

To further decorate the stratification, we need some notation. Write

$$\vec{a} = \big((a_1, \varepsilon_1), \dots, (a_k, \varepsilon_k) \big)$$

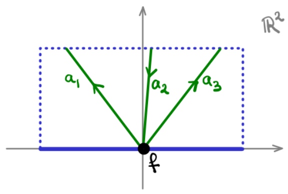
for a list of tuples with $a_i \in S$ and $\varepsilon_i \in \{\pm\}$. We extend the functor $T : S^{\mathrm{str}} \to S$ to act on such tuples by declaring $a^+ = a$ and $a^- = a^*$, so that

$$T(\vec{a}) = a_1^{\varepsilon_1} \otimes (a_2^{\varepsilon_2} \otimes \cdots (a_{k-1}^{\varepsilon_{k-1}} \otimes a_k^{\varepsilon_k}) \cdots) .$$

Given $\vec{a}$ of length k, we decorate H_k as follows:

- Label the i'th ray from the left by a_i, and orient it to point away from 0 for $\varepsilon_i = +$ and towards 0 if $\varepsilon_i = -$.
- Label the 0-stratum at 0 by a morphism $f : \mathbb{1} \to T(\vec{a})$ in S.

Write $H_k(\vec{a}, f)$ for the resulting stratified 2-manifold. For example, for $\vec{a} = ((a_1, +), (a_2, -), (a_3, +))$ have $f : \mathbb{1} \to a_1 \otimes (a_2^* \otimes a_3)$ and

$H_3(\vec{a}, f) =$ 

As an aside, the above picture is why we did not use the partial order $x < s(x)$, $x < t(x)$ for all $x \in D_0$, with $s, t : D_0 \to D_1$ the source and target of a morphism, as we did above Definition 3.14? The reason is that in the above example this would require each of a_1, a_2, a_3 to be either $s(f)$ or $t(f)$, which is not what we want.

Definition 4.23. Let S be a rigid symmetric monoidal category.

1. An *n-dimensional defect bordism for $D(S)$* is a $D(S)$-stratified n-dimensional (oriented) bordism $M : X \to Y$ such that all 1-strata in M (and 0-strata in X, Y) are in addition oriented, and such that each 0-stratum v in the interior of M is equipped with an embedding $\varphi_v : H_k(\vec{a}; f) \to M$ for some $\vec{a}, f$, which maps $0 \in H_k$ to $v \in M$ and which respects the stratification and orientations of the 1-strata (but of course does not map the boundary of H_k to the boundary of M as v is in the interior).

2. Two such bordisms are isomorphic if they are isomorphic as stratified bordisms, and the isomorphism respects orientations of the 1-strata and the embeddings φ_v.

3. The bordism category $\mathsf{Bord}_n^{\mathrm{def}}(S)$ has as objects closed $D_{\geq 1}$ stratifed $(n-1)$ manifolds, such that the 0-strata are oriented and equipped with a total order, that is, a map $\{1, 2, \ldots, |X_0|\} \to X_0$. Morphisms are isomorphism classes of bordisms as in part 1.

Now we construct a symmetric monoidal functor

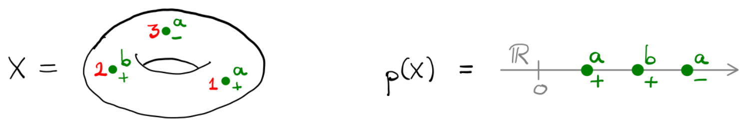
$$L : \mathsf{Bord}_n^{\mathrm{def}}(S) \to S$$

by "forgetting the n-manifold and evaluating the 1-strata via a 1-dimensional TFT." (L stands for "line".) (6.5.26)

■ The 1-dimensional TFT in question is $Z : \mathsf{Bord}_1^{\mathrm{def}}(D_1) \to S$, cf. Definition 3.14, where $D_1 = \mathrm{obj}(S)$ as before, and there is no D_0 here (that is, $D_0 = \emptyset$). Correspondingly, there are no 0-strata. The TFT Z is uniquely determined by setting its values on the a-labelled $\pm$-oriented points to be

$$Z(\bullet_+^a) = a \quad , \quad Z(\bullet_-^a) = a^* \ .$$

■ Now we extract an object in $\mathsf{Bord}_1^{\mathrm{def}}(D_1)$ from an object $X \in \mathsf{Bord}_n^{\mathrm{def}}(S)$. By assumption, the 0-strata are totally ordered, and we can collect their D_1-labels and orientations into a standard set $p(X)$ of oriented S-labelled points, $\{1, 2, \dots, |X_0|\} \subset \mathbb{R}$. For example:

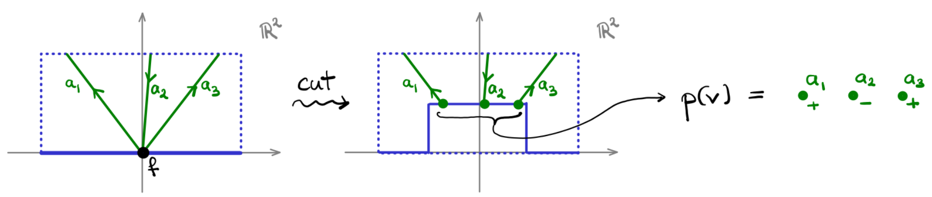


We define

$$L(X) = Z(p(X)) = a_1^{\varepsilon_1} \otimes (a_2^{\varepsilon_2} \otimes \cdots (a_{m-1}^{\varepsilon_{m-1}} \otimes a_m^{\varepsilon_m}) \cdots) \in S \ .$$

In the example above, $L(X) = a \otimes (b \otimes a^*)$.

■ For morphisms $M : X \to Y$ in $\mathsf{Bord}_n^{\mathrm{def}}(S)$ the procedure is more complicated due to the zero strata. For a 0-stratum $v \in M_0$, we use the parametrisation $\varphi_v : H_k(\vec{a}; f) \to M$ to "cut out the half-disc at radius $\frac{1}{2}$ around 0" and in this way create an object $p(v) \in \mathsf{Bord}_1^{\mathrm{def}}(D_1)$. For example,

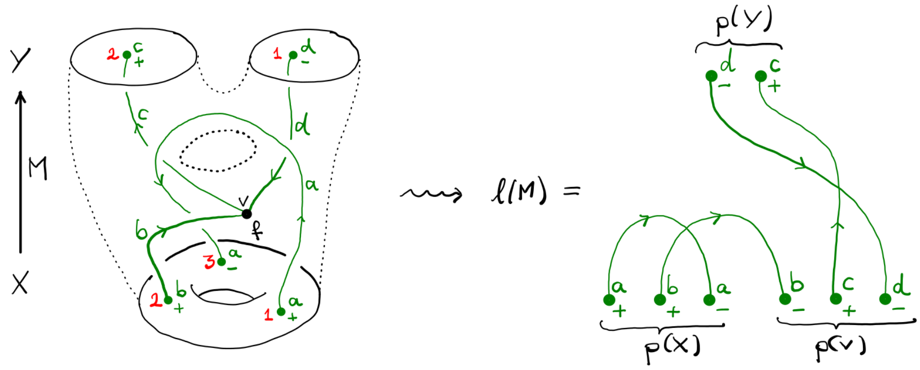


By construction, $Z(p(v)) = T(\vec{a})$. (Why?) ?

Pick an order $v_1, \dots, v_{|M_0|}$ of the 0-strata in M. Then the 1-strata, with the part near the 0-strata removed, now form a bordism

$$\ell(M) : p(X) \sqcup \bigsqcup_{i=1}^{|M_0|} p(v_i) \longrightarrow p(Y)$$

in $\mathsf{Bord}_1^{\mathrm{def}}(D_1)$. For example,

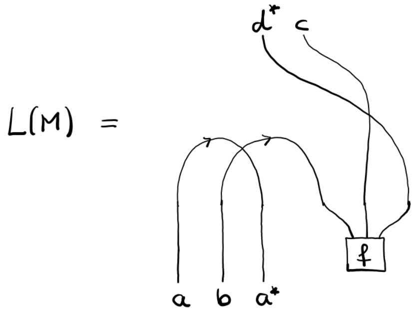


Here, the top boundary shows Y and not $-Y \subset \partial M$. For $-Y$ the 0-strata would be labelled $\bullet_{-}^{c}$ and $\bullet_{+}^{d}$.

■ Let $v \in M_0$ be a 0-stratum with parametrisation $\varphi_v : H_k(\vec{a}; f_v) \to M$ as above. By definition, $f_v : \mathbb{1} \to T(\vec{a}) = Z(p(v))$, and we can use the f_v to obtain a morphism $Z(p(X)) \to Z(p(Y))$ from $\ell(M)$. This defines our functor L on morphisms (we state $L(M)$ for S strict to not make the expression too long):

$$L(M) = \Bigg[L(X) = Z\big(p(X)\big) \xrightarrow{1 \otimes \bigotimes_i f_{v_i}} Z\big(p(X)\big) \otimes \bigotimes_{i=1}^{|M_0|} Z\big(p(v_i)\big)$$
$$\xrightarrow{\sim} Z\Big(p(X) \sqcup \bigsqcup_{i=1}^{|M_0|} p(v_i)\Big) \xrightarrow{Z\left(\ell(M)\right)} Z\big(p(Y)\big) = L(Y) \Bigg]$$

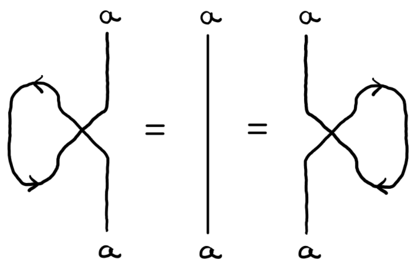
In the example above, we find the following morphism in S, expressed in graphical calculus,



One can check from the symmetry of S that $L(M)$ is independent of the choice of ordering $v_1, \ldots, v_{|M_0|}$ of the 0-strata in M. It is functorial and symmetric monoidal because Z is. Altogether we have:

Proposition 4.24. Let S be a rigid symmetric monoidal category. Then $L : \mathsf{Bord}_n^{\mathrm{def}}(S) \to S$ as defined above is a symmetric monoidal functor.

Exercise 4.25. Let S be a rigid symmetric monoidal category. Recall from Section 2.3 that if a^* is the left dual of $a \in S$, then a^* is also a right dual with, for example $\widetilde{\mathrm{ev}}_a = \mathrm{ev}_a \circ \beta_{a,a^*} : a \otimes a^* \to \mathbb{1}$. Check by direct computation from properties of the braiding that E

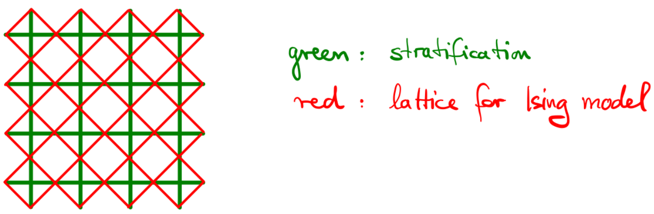


4.5 State sum models

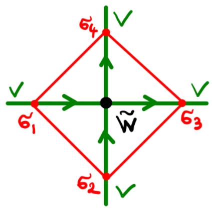
Two-dimensional Ising lattice model

As an example of the trivial TFT with line defects introduced in the previous section, let us write the 2d Ising lattice model as the value of $L : \mathsf{Bord}_2^{\mathrm{def}}(\mathsf{Vec}_{\mathbb{C}}) \to \mathsf{Vec}_{\mathbb{C}}$ on a specific bordism.

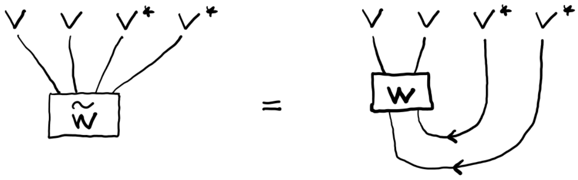
Recall from Example 3.16 that the Ising lattice model as a variable ± 1 – the "lattice spin" – assigned to a vertex, or "site" of the lattice, while the TFT description encodes these as basis of the vector space $V = \mathbb{C}^2$ assigned to edges. We can match these descriptions by rotating the lattice by 45°:



Here all 1-strata are labelled by $V = \mathbb{C}^2$. To find the label $\tilde{W}$ of the 0-strata,



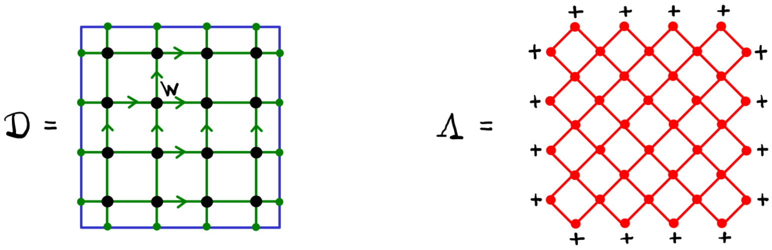
we write the weights $e^{-\beta E(\sigma)}$ in the Ising partition function as a product over weights assigned to 0-strata, where for each 0-stratum we collect the contribution of the four Ising model edges surrounding it. It is convenient to replace the label $\tilde{W} : \mathbb{C} \to V \otimes V \otimes V^* \otimes V^*$ by a map $W : V \otimes V \to V \otimes V$ via the coevaluation for V,



As in Example 3.16, write $e_+ = (1,0)$, $e_- = (0,1)$ for the standard basis of V, and $e^*_\pm \in V^*$ for the dual basis. The linear map W is defined on a basis as, for $\sigma_{1,2,3,4} \in \{\pm\}$,

$$(e^*_{\sigma_4} \otimes e^*_{\sigma_3}) W(e_{\sigma_1} \otimes e_{\sigma_2}) = \exp\big(\beta(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1)\big) \ .$$

For example, take



Here D is a stratified solid square, understood as a bordism from the stratified boundary square to the empty set. We labelled the 0-strata in the interiour by W, but it is understood that this is just a placeholder for $\tilde{W}$ expressed in terms of W. (What does that look like?) The boundary has eight 0-strata $\bullet^V_+$ and eight 0-strata $\bullet^V_-$. ?

Λ is the lattice on which to compute the Ising model partition function, and the "+" indicate that in the sum over configurations, these lattice spins are fixed as "+".

The TFT L evaluated on D computes the Ising model partition function:

$$L(D)\big((e_+)^{\otimes 8} \otimes (e^*_+)^{\otimes 8} \big) \ = \ Z^{\text{Ising}}_\Lambda(\beta) \ .$$

(If you would write out the sum over configurations in $Z^{\text{Ising}}_\Lambda(\beta)$ explicitly, how many summands would it have? How would you describe the "free" boundary condition in the Ising model on the TFT side? That is, the boundary spins in Λ are not fixed, but also summed over.) ?

As in the 1d case in Example 3.16, we get a functorial formulation of the 2d Ising lattice model in terms of functors out of a category of stratified 2d bordisms.

Topological state sum model

The Ising lattice model is non-topological in the sense that it depends on the choice of stratification. As in the 1d case we can ask if for specific label choices for 1- and 0-strata the value of L becomes independent of the stratification (within a suitable class of stratifications).

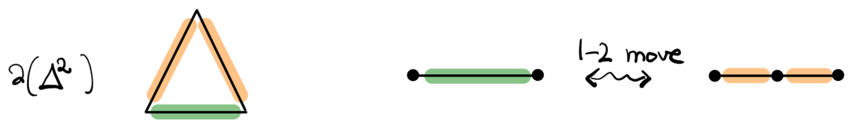
■ An idea which can be formulated in any dimension is to triangulate a bordism and then show that the value of L is independent of the triangulation. This can be done by showing invariance under *bistellar flips* or *Pachner moves*:

Definition 4.26. Let M be an n-manifold with triangulation T. Let $S \subset T$ be an n-dimensional subcomplex (a union of n-simplices) which is topologically a disc, and let $f : S \to \partial(\Delta^{n+1})$ a simplicial embedding (i.e. injective and maps simplices to simplices). Then we say T' is obtained from T by a *bistellar flip* or *Pachner move* if removing S in T and replacing it with its complement in $\partial(\Delta^{n+1})$ results in T':

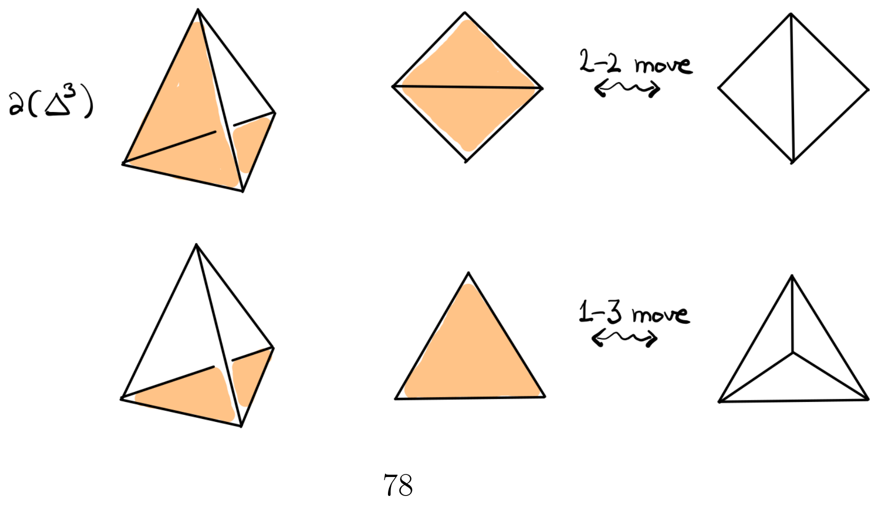
$$T' = \big(T \setminus \mathrm{int}(S)\big) \sqcup_{f|_{\partial S}} \big(\partial\Delta^{n+1} \setminus \mathrm{int}(f(S))\big) \ .$$

Example 4.27.

1. $n = 1$: There is just one way to split $\partial(\Delta^2)$ into two disc-shaped sub-complexes, and so there is one 1d Pachner move (and its inverse), the 1-2 move:



2. $n = 2$: There are two ways to split $\partial(\Delta^3)$ and we get two 2d Pachner moves, the 2-2 move and the 1-3 move:



3. There are two 3d Pachner moves (the 2-3-move and the 1-4-move), and three 4d Pachner moves (3-3, 2-4, 1-5).

We import the following result on existence of triangulations and how to related triangulations via Pachner moves:

Proposition 4.28. Let M be an n-manifold and T_∂ a triangulation of ∂M. Then

1. there exists a triangulation T of M such that $T \cap \partial M = T_\partial$, and

2. any two triangulations as in 1. are related by a finite sequence of Pachner moves.

For part 1 we already know from the PL-structure of M that triangulations exist. A fixed boundary triangulation can be extended by choosing a collar neighbourhood of the boundary and first triangulating that in a way that respects T_∂ (see [RSa, Cor. 2.26] for the existence of such collars, and [Mun, Thm. 10.6] for this construction of the extension of T_∂ in the context of differentiable manifolds). Part 2 can be found in [Ca], which also references the original paper by Pachner for the non-boundary case.

Note that by construction, Pachner moves leave the triangulation T_∂ of the boundary invariant. (Why?) ?

■ Let M be an n-manifold and T a triangulation. Below we will use oriented stratifications, and we can encode these orientations by a total order O_T on the vertices of the triangulation. This gives an orientation to each k-simplex Δ^k in T by mapping it to the k-simplex with vertices $0, e_1, \dots, e_k \in \mathbb{R}^k$ (zero, together with the standard basis vectors) according to the order over vertices in Δ^k. This orients Δ^k via the standard orientation of $\mathbb{R}^k$. We call (T, O_T) an *oriented triangulation*. One way to describe a total order is to assign distinct real numbers ("heights") to each vertex of T.

■ We can assemble bordisms with oriented triangulations into a semicategory. A *semicategory* (or *non-unital category*) is simply what is left from a category if one drops the requirement that there are identity morphism $1_a : a \to a$, as well as the corresponding axioms for the composition. That is, a semicategory consists of objects and morphisms, and has an associative composition on these.

Definition 4.29. The semicategory Bord_n^T of *bordisms with oriented triangulations* has

- objects: triples $X = (X, T_X, O_X)$ where $X \in \mathsf{Bord}_n$, and (T_X, O_X) is an oriented triangulation of X.

- morphisms $X \to Y$: triples (M, T_M, O_M) where $M : X \to Y$ is a morphism in Bord_n, (T_M, O_M) is an oriented triangulation of M such that the embeddings $X, Y \to \partial M$ preserve the triangulation and the vertex ordering. (The embedding of Y reverses orientation by definition, but we keep the vertex ordering the same.) We require that the total order O_M satisfies

$$(\text{vertices of } T_Y) > (\text{inner vertices of } T_M) > (\text{vertices of } T_X) \ .$$

- Composition of $X \xrightarrow{M} Y$ and $Y \xrightarrow{N} Z$ is by gluing, with induced triangulation T on $N \circ M$, and ordering

$$\begin{aligned}(\text{vertices of } T_Z) &> (\text{inner vertices of } T_N) > (\text{vertices of } T_Y)\\ &> (\text{inner vertices of } T_M) > (\text{vertices of } T_X)\end{aligned}$$

(Why is Bord_n^T only a semicategory, i.e. why does it not have identity morphisms?) ?

For info: We can define a tensor product functor on Bord_n^T by disjoint union and an appropriate ordering. However, as in the current definition Bord_n^T has no identity morphisms, we cannot define coherence isomorphisms or a braiding isomorphism. Thus we cannot make Bord_n^T symmetric monoidal. One way out is to enlarge the morphisms in Bord_n^T to contain isomorphisms between objects, in addition to triangulated bordisms. This works, but is again more technical, and we will make do without a symmetric monoidal structure on Bord_n^T.

■ Let $U : \mathsf{Bord}_n^T \to \mathsf{Bord}_n$ be the functor which forgets the oriented triangulation. Thus, a TFT $Z \in \mathsf{TFT}_n(S)$ defines a functor out of Bord_n^T via

$$\mathsf{Bord}_n^T \xrightarrow{U} \mathsf{Bord}_n \xrightarrow{Z} S \ .$$

We are interested in the opposite question: Given $Y : \mathsf{Bord}_n^T \to S$, can we find Z in

$$\begin{tikzcd} \mathsf{Bord}_n^T \arrow[r, "Y"] \arrow[d, "U"'] & S \\ \mathsf{Bord}_n \arrow[ur, dashed, "Z"'] & \end{tikzcd}$$

such that the above diagram "commutes" in a suitable sense?

Note that we will typically not want Y and $U \circ Z$ to be naturally isomorphic or even equal. E.g. in the 1d state sum construction (Section 3.3), from Y we obtained an idempotent p on $a = Y(\bullet_+)$, and we defined $Z(\bullet_+) = j$, the image of p. (In Section 3.3, Y was called Z, and Z was called Z^{new}.) (20.5.26)

For info: Triangulations are not convenient to work with in practice because one needs too many triangles, making explicit computations difficult. One can formulate state-sum constructions for more general cell complexes, where the individual cells are not only simplicies but can be more general ball-shaped polyhedra. These decompositions are called PLCW-complexes and were introduced with topological state-sum models in mind, see [Ki1].

■ We now focus on two dimensions. Fix a rigid symmetric monoidal category S. We assume that left- and right duals are related via the symmetric braiding (as above Proposition 2.33),

$$\widetilde{\mathrm{ev}}_a = \mathrm{ev}_a \circ \beta_{a,a^*} : a \otimes a^* \to \mathbb{1} \quad , \quad \widetilde{\mathrm{coev}}_a = \beta_{a,a^*} \circ \mathrm{coev}_a : \mathbb{1} \to {}^*a \otimes a \ .$$

We will build a functor

$$D = D_{A,\mu,\Delta,\psi} : \mathsf{Bord}_2^T \to \mathsf{Bord}_2^{\mathrm{def}}(S)$$

which depends on a choice of

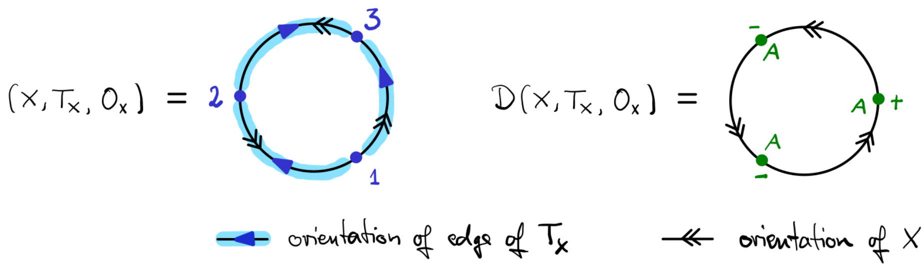
$$A \in S \quad , \quad \mu : A \otimes A \to A \ , \quad \Delta : A \to A \otimes A \ , \quad \psi : \mathbb{1} \to A \ .$$

Roughly speaking, D takes the Poincaré dual of the triangulation, decorates 1-strata by A and 0-strata by μ or Δ, and assigns a weight ψ to each vertex. Then we will look for conditions on μ, Δ, ψ so that

$$Y = \left[\mathsf{Bord}_2^T \xrightarrow{D_{A,\mu,\Delta,\psi}} \mathsf{Bord}_2^{\mathrm{def}}(S) \xrightarrow{L} S \right]$$

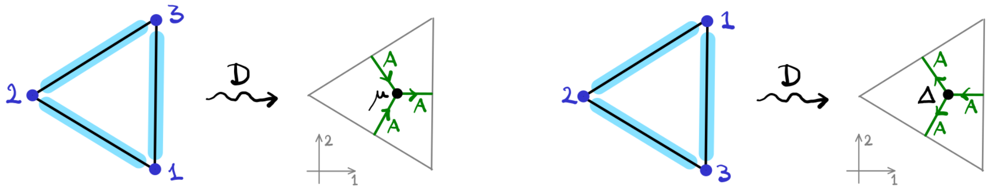
is triangulation-independent. In more detail, the definition of D is as follows.

- Let
$$X = (X, T_X, O_X) \in \mathsf{Bord}_2^T \ .$$
On each 1-simplex of T_X mark the mid-point as 0-stratum, label it A and orient it "+" if the orientation of the 1-simplex agrees with that of the 1-manifold X, and "−" otherwise.[5] For example,



[5]To match Definition 4.23 (3), we also need a give total order of the 0-strata. We will not use this explicitly below, but one way to get such an ordering is to assign to each 0-stratum the first vertex encountered when walking along X in the direction of its orientation, and take the ordering induced by O_X.

- Let $M : X \to Y$ be a morphism in Bord_2^T. For each triangle in T_M, depending on the ordering of the vertices we choose the stratification

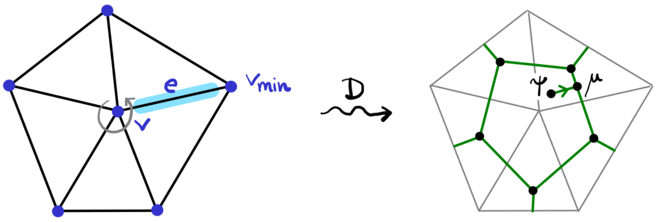


The triangles are shown such that their orientation as induced by M agrees with the paper-plane orientation, as indicated by the little coordinate system. (So for one of the triangles the orientation obtained from the vertex order agrees with that of M, and for one it does not. Which is which?) ?

As in the Ising model example, μ, Δ are placeholders for $\tilde{\mu} : \mathbb{1} \to A \otimes (A^* \otimes A^*)$, $\tilde{\Delta} : \mathbb{1} \to A \otimes (A \otimes A^*)$.

Note that thanks to the total order, the case where all three 1-strata point towards the 0-stratum in the centre of a triangle, or all point away from it, cannot happen. (Why?) ?

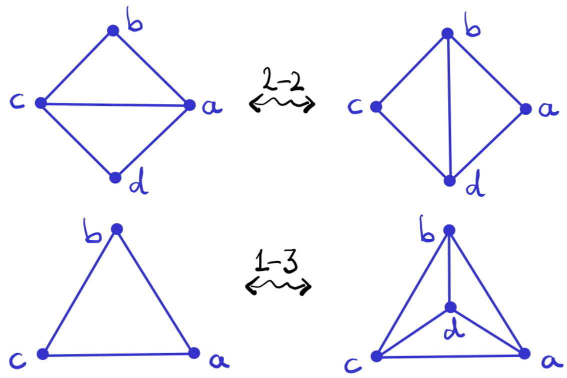
Finally, let v be a vertex of T_M not contained in T_Y, i.e. not contained in the outgoing boundary component Y of M. Pick the smallest vertex v_{min} (in the order O_M) connected to v by an edge e. Place a 0-stratum ψ in the triangle located counter-clockwise from the edge e relative to the vertex v and the orientation of M:



The orientations of the 1-strata on the right hand side that are not indicated depend on the vertex ordering. Note that we can label the vertex where the 1-stratum starting at ψ meets the rest of the network of A-lines by μ, independent of the vertex ordering. (Why?) ?

(The rule of not including ψ's for vertices in the outgoing boundary is necessary for functoriality. Why is that?) ?

■ By *oriented 2d Pachner moves* we mean the usual 2d Pachner moves, but where keep track of the total order of the vertices,



where $a, b, \dots \in \mathbb{R}$ are heights used to encode the total order. In the 1-3 move, in the direction from left to right we add a new vertex, and we can choose its height d arbitrarily as long as

$$(\text{vertices of } T_Y) > (\text{inner vertices of } T_M) > (\text{vertices of } T_X) \ .$$

is not violated.

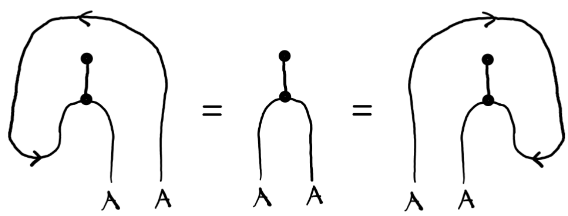
E

Exercise 4.30. Show that the oriented Pachner moves allow one to change the ordering O_M of a given triangulation T_M arbitrarily (subject to the above condition that inner vertices lie between the vertices on the in- and outgoing boundary). *Hint:* For an inner vertex v, apply 2-2 moves in the neighbourhood of v until the 1-3 move can be applied to remove and re-insert a vertex.

Via the functor D, the oriented Pachner moves get translated into identities for μ, Δ, ψ. We will not try to analyse these systematically[6] and instead will make an ansatz and check that it works.

■ We need two more property of Frobenius algebras:

Definition 4.31. A Frobenius algebra A in a rigid symmetric monoidal category S (with left and right dualities related by the symmetric braiding) is called *symmetric* if (in S^{str})



[6]For info: We could of course do that, but for that it would be better to not determine the point where ψ is attached by the total order O_M. Instead we should make that an extra datum which is a map $\{\text{vertices}\} \to \{\text{triangles}\}$ assigning one of the triangles sharing the vertex v as the one to contain the 0-stratum ψ. We would find (in particular) that (A, μ, Δ, ψ) defines a non-unital, non-counital Frobenius algebra.

In fact, the two equalities above are equivalent, as each one can be rewritten as (Why? *Hint:* Use Exercise 4.25.)

$$\varepsilon \circ \mu \circ \beta_{A,A} = \varepsilon \circ \mu \ ,$$

i.e. the non-degenerate pairing is symmetric.

(For info: The reason we start from the more complicated condition is that it is still correct in pivotal categories. These are briefly described in the info-point on page 114. On the other hand, the above reformulation does not require rigidity, and so is useful in non-rigid symmetric monoidal categories.)

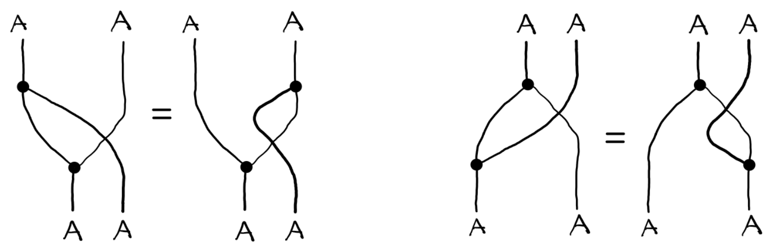
Example 4.32. Let $S = \mathsf{Vec}_k^{\mathrm{fd}}$ be the category of finite-dimensional vector spaces over a field k. Let A be a Frobenius algebra in S, that is, a Frobenius algebra over k. By definition, A is equipped with a non-degenerate pairing $\langle a, b\rangle = \varepsilon(ab)$, for $a, b \in A$. Then A is symmetric if and only if $\langle a, b\rangle = \langle b, a\rangle$ for all $a, b \in A$, that is, if the pairing is symmetric. (Can you give an example of a non-symmetric Frobenius algebra?)

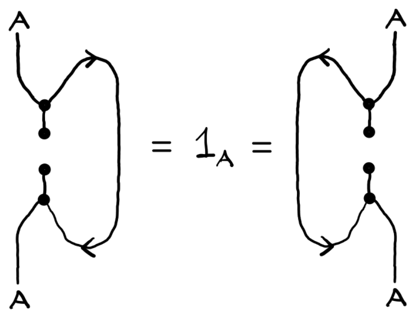
Exercise 4.33. Let S be a rigid symmetric monoidal category and A a symmetric Frobenius algebra in S. Show that also

$$\beta_{A,A} \circ \Delta \circ \eta = \Delta \circ \eta \ .$$

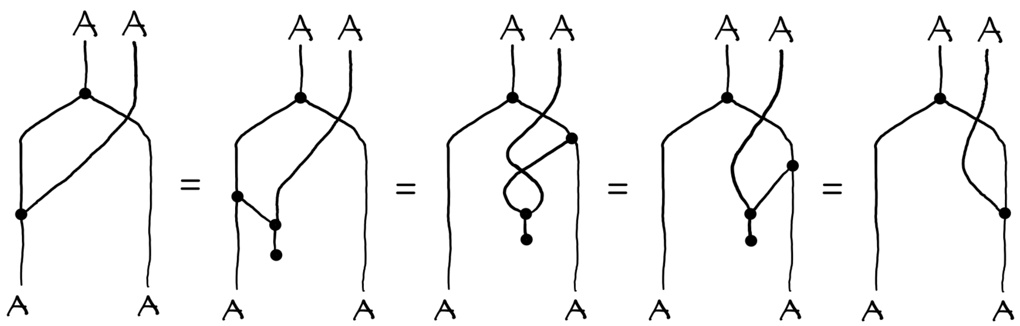
The following lemma collects some useful identities that hold for symmetric Frobenius algebras.

Lemma 4.34. Let A be a symmetric Frobenius algebra in a symmetric rigid monoidal category. Then

1. 

2. 

Proof. For part 1, the second equality amounts to the computation

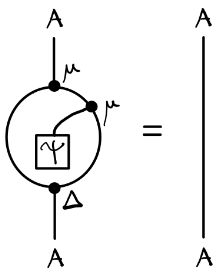


The first equality in part 1 can be seen similarly. To see the identities in part 2, compose with the pairing $\varepsilon \circ \mu$ and note that this produces the identities in Definition 4.31 satisfied by a symmetric Frobenius algebra. (Details?) $\square$?

Definition 4.35. A Frobenius algebra A in a monoidal category C is called *separable with separability element* ψ if

$$\left[A \xrightarrow{\Delta} AA \xrightarrow{\sim} A(\mathbb{1}A) \xrightarrow{1_A\otimes(\psi\otimes 1_A)} A(AA) \xrightarrow{1_A\otimes\mu} AA \xrightarrow{\mu} A\right] \;=\; 1_A\ .$$

In graphical calculus in C^{str},



Exercise 4.36. Let A be a Frobenius algebra in a monoidal category C. Show that A is separable if and only if (E)

$$\left[\mathbb{1} \xrightarrow{\eta} A \xrightarrow{\Delta} AA \xrightarrow{\sim} A(\mathbb{1}A) \xrightarrow{1_A\otimes(\psi\otimes 1_A)} A(AA) \xrightarrow{1_A\otimes\mu} AA \xrightarrow{\mu} A\right] \;=\; \eta\ .$$

i.e. if and only if the condition obtained from that in Definition 4.35 when precomposed with $\eta : \mathbf{1} \to A$ holds.

Example 4.37. Let us see if the Frobenius algebras discussed in Example 4.8 are separable or not.

1. k with $\Delta(1) = \delta^{-1}1 \otimes 1$ is separable with separability element $\psi = \delta$.

2. $\mathrm{Mat}_n(K)$ for a commutative ring K is separable,[7] for example we can choose $\psi = E_{11}$, the elementary matrix with a 1 in the top left corner. Let $I = \sum_{i=1}^{n} E_{ii}$ be the unit matrix. As

$$\Delta(I) = \sum_{i,j=1}^{n} E_{ij} \otimes_K E_{ji} \ ,$$

the left hand side of the separability condition in Exercise 4.36 becomes

$$\sum_{i,j=1}^{n} E_{ij} E_{11} E_{ji} = \sum_{i=1}^{n} E_{i1} E_{1i} = \sum_{i=1}^{n} E_{ii} = I \ .$$

(Use this example to argue that the separability element is in general not unique.) ?

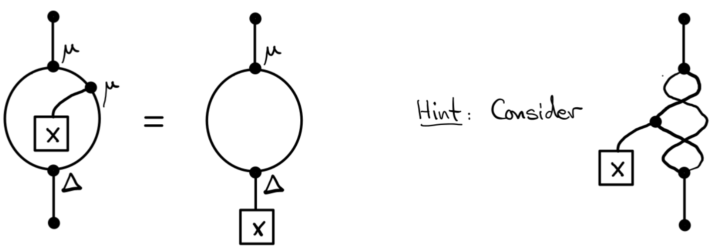
3. For the group algebra $k[G]$, with G a finite group, suppose that $\mathrm{char}(k)$ does not divide $|G|$. Then $|G| \neq 0$ in k and we can take $\psi = |G|^{-1}$. (Check that this works.) ?

4. $k[X]/\langle X^d \rangle$ is not separable for $d > 1$. Indeed, let ψ be arbitrary. We have $\Delta(1) = \sum_{k=0}^{d-1} X^k \otimes X^{d-1-k}$, and the condition in Exercise 4.36 becomes

$$\sum_{k=0}^{d-1} X^k \psi X^{d-1-k} = d X^{d-1} \psi \ .$$

But dX^{d-1} has no multiplicative inverse in $k[X]/\langle X^d \rangle$, so the above expression can never be 1.

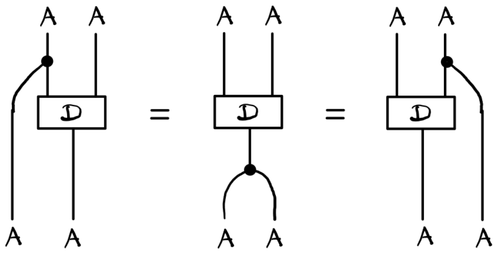
Exercise 4.38. 1. For A a symmetric Frobenius algebra in a rigid symmetric monoidal category S, and any $x : \mathbb{1} \to A$, show that E



[7]One choice for S such that $\mathrm{Mat}_n(K)$ is a separable symmetric Frobenius algebra in S is that of finite-rank free K-modules, as these form a rigid category. One advantage of the alternative formulation of symmetry just below Definition 4.31 is that it does not need S to be rigid and one can just take $S = K\text{-}\mathsf{mod}$.

2. Show that for a finite group G, if $|G| = 0$ in a field k, then $k[G]$ is not separable. *Hint:* Assume there exists a separability element and bring this to a contradiction using part 1 of the exercise.

Exercise 4.39. *(Separable algebras)* Let C be a monoidal category and A an algebra in C. A is called *separable* if there is $D : A \to A \otimes A$ which satisfies (in C^{str}): E



Equivalently, D is an A-A-bimodule morphism, but we did not define these yet. Furthermore, D must satisfy

$$\mu \circ D = 1_A \ .$$

For $C = \mathsf{Vec}_k$ this is the standard definition of a separable algebra.

The aim of this exercise is to show that if A is Frobenius, then it is separable as an algebra if and only if there is a separability element as in Definition 4.35. Let thus A be a Frobenius algebra in C.

1. Suppose there is an element D as above. Show that $\psi = \left[\mathbb{1} \xrightarrow{\eta} A \xrightarrow{D} AA \xrightarrow{\varepsilon \otimes 1_A} \mathbb{1}A \xrightarrow{\sim} A\right]$ is a separability element for A.

2. Let ψ be a separability element for A. Show that there is D which makes A separable as an algebra in the above sense.

For info: For info: One important consequence of separability of an algebra A is, in the case $C = \mathsf{Vec}_k$, that the category of A-modules is semisimple. Indeed, one can use D to write any module M as a retract of the free module $A \otimes_k M$, and so every module is projective.

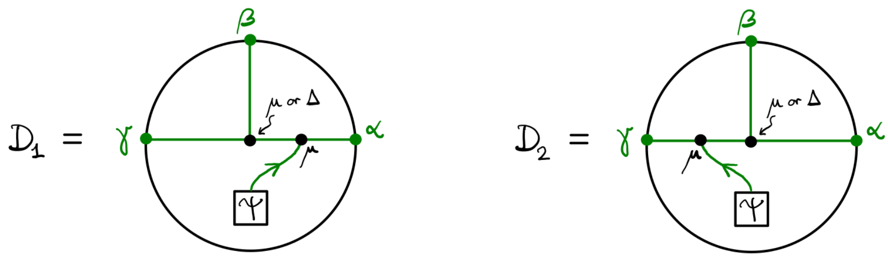
■ From hereon we assume:

> $A = (A, \mu, \eta, \Delta, \varepsilon)$ is a separable symmetric Frobenius algebra in S with separability element ψ.

With this, we will establish in a series of lemmas that $Y = L \circ D : \mathsf{Bord}_2^T \to S$ is independent of the choice of oriented triangulation (T_M, O_M) of a bordism $M : X \to Y$ in Bord_2^T (with fixed oriented triangulation of X, Y). We will do this by first showing some identities for L evaluated on disc-shaped bordisms with A-labelled defect networks, and then turn to Y.

The first lemma will allow us to "move ψ's out of the way" by showing that we move the point where ψ is attached past the 0-startum labelled μ or Δ in the centre of a triangle:

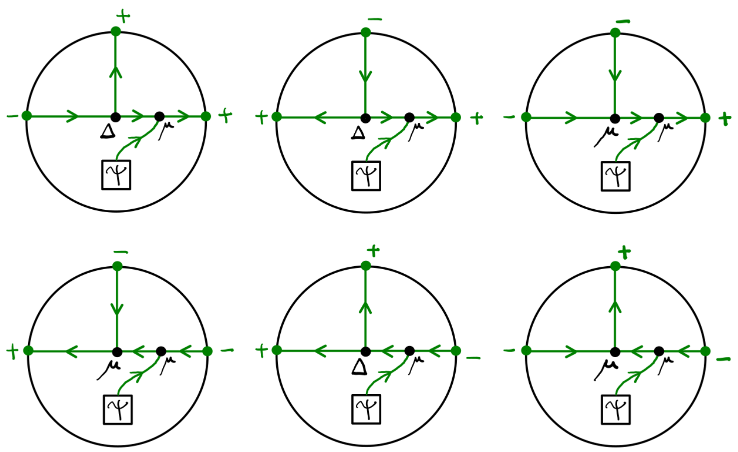
Lemma 4.40. Consider the morphisms $D_1, D_2 : \emptyset \to S^1$ in $\mathsf{Bord}_2^{\mathrm{def}}(S)$,



where $\alpha, \beta, \gamma \in \{\pm\}$ are not all "+" and not all "$-$". Then

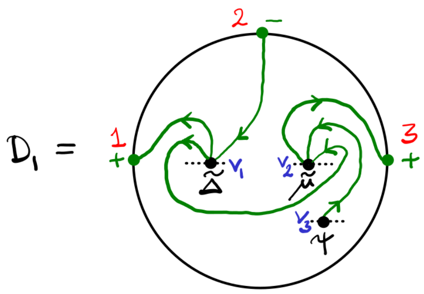
$$L(D_1) = L(D_2) \ .$$

Proof. There are six possibilities for the orientations α, β, γ. For D_1 this gives

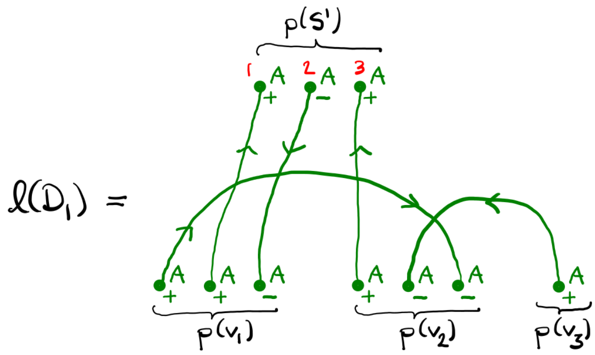


The corresponding identity follows from associativity and Frobenius property (try a few), except for the second case. So let us go through the computation for the second case in some detail as an example, and be more brief in the following proofs.

We need to compute $L(D_1)$ via the procedure given in Section 4.4, for D_1 the second disc shown above. We do this in steps. First we undo our convention of labelling 0-strata by a morphism with in- and outgoing legs and instead use $\tilde{\mu} : \mathbb{1} \to A \otimes A^* \otimes A^*$ and $\tilde{\Delta} : \mathbb{1} \to A \otimes A \otimes A^*$:



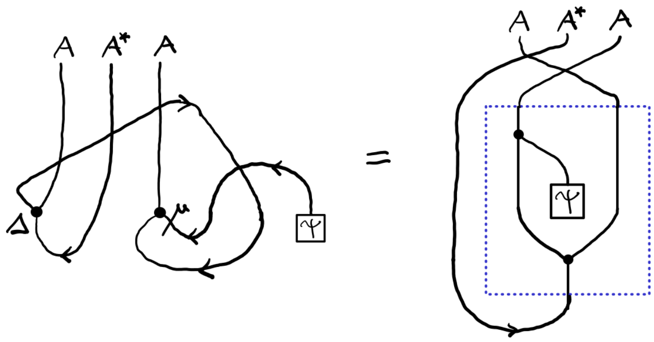
Let us agree that the ordering of the boundary and interior 0-strata is as indicated. Then $p(v_1) = \bullet^A_+ \sqcup \bullet^A_+ \sqcup \bullet^A_-$, etc. The bordism $\ell(D_1)$ is



From there we can compute $L(D_1)$ as

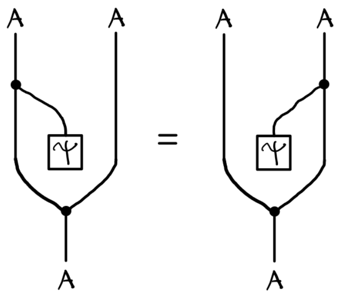
$$L(D_1) = \left[\mathbb{1} \xrightarrow{\tilde{\Delta}\otimes\tilde{\mu}\otimes\psi} AAA^*AA^*A^*A \xrightarrow{Z^{1\mathrm{d}}(\ell(M))} AA^*A \right] ,$$

where $Z^{1\mathrm{d}}$ is the 1d TFT used in Section 4.4. As a string diagram, the result is, with $\tilde{\Delta}$ and $\tilde{\mu}$ replaced again by Δ, μ:

$L(D_1) =$ 

In the second expression we deformed the diagram. The morphism in the dashed box is the left hand side in the chain of equalities of morphisms in S

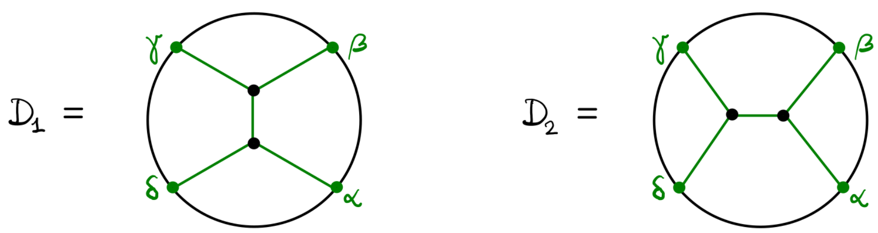
(in string diagram notation) below. If we repeat the above computation for D_2, arranging the answer so that the diagram is the same outside the dashed box, then the diagram inside the box is the right hand side of the following equality:



This equality follows from composing the first equality in Lemma 4.34 with ψ. Altogether this shows that $L(D_1) = L(D_2)$. $\square$

The next lemma will be used in the 2-2 Pachner move. The identity is sometimes called "fusion move".

Lemma 4.41. *Consider the morphisms* $D_1, D_2 : \emptyset \to S^1$ *in* $\mathsf{Bord}_2^{\mathrm{def}}(S)$,



where $\alpha, \beta, \gamma, \delta \in \{\pm\}$ *are not all "+" and not all "−" and the inner edge can be oriented any way such that the three-valent vertices can be labelled by either* μ *or* Δ. *Then*

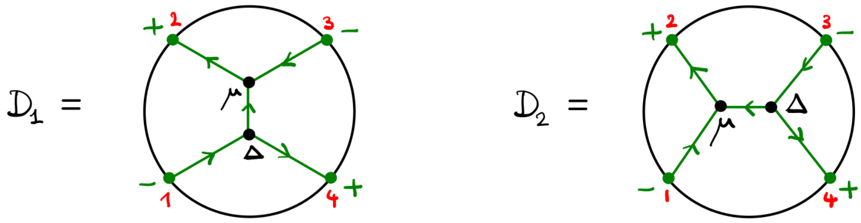
$$L(D_1) = L(D_2) \ .$$

Proof. There are quite a few identities to check, and we will just look at a few.

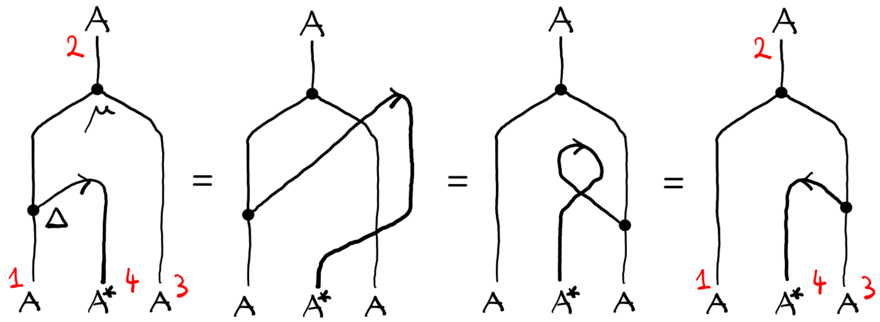
For example, if $\beta = +$, $\alpha = \gamma = \delta = -$ with inner edge in D_1 pointing up and inner edge in D_2 pointing to the right (In both cases this is the only choice. Why?), all inner 0-strata are labelled μ and the resulting identity is associativity of A. ?

If $\alpha = \delta = -$ and $\beta = \gamma = +$, then for D_1 the inner edge has to point upwards, while for D_2 one has two choices. This results in two identities which are precisely the Frobenius relations. (Check this.) ?

A more complicated case is $\alpha = \gamma = +$ and $\beta = \delta = -$, with inner edges oriented as shown:



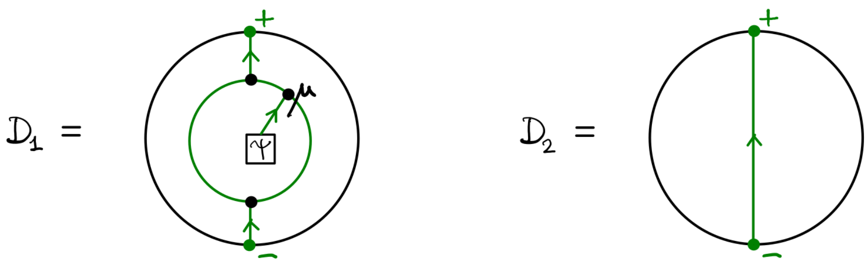
The numbers $1, 2, 3, 4$ indicate the ordering of the boundary 0-strata. Following the same procedure as in the proof of Lemma 4.40 one computes the string diagrams for the morphisms $L(D_1), L(D_2) : \mathbb{1} \to A^* \otimes A \otimes A^* \otimes A$, and then isolates a convenient part of the diagram on which to show the equality. This amounts to the calculation shown below (we label the objects by 1,2,3,4 in the same order as the boundary 0-strata):



In the second step we apply the second equality in Lemma 4.34 (1) and deform the diagram. The last equality we substitute $\widetilde{\mathrm{ev}}_A \circ \beta_{A^*,A} = \mathrm{ev}_A$. $\square$

The next two lemmas will be used for the 1-3 move. The identity in this lemma is also called "bubble move".

Lemma 4.42. Consider the morphisms $D_1, D_2 : \emptyset \to S^1$ in $\mathsf{Bord}_2^{\mathrm{def}}(S)$,



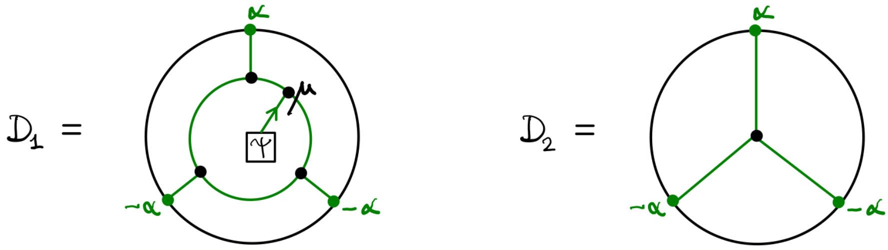
where the inner edges can be oriented in any way such that the three-valent vertices can be labelled by either μ or Δ. Then

$$L(D_1) = L(D_2) \ .$$

Exercise 4.43. There are three possibilities for orienting the inner edges. Show two of the resulting three identities. E

In the proof of the following lemma we will use the technique of "cutting out a disc and gluing back another disc" which will also be instrumental in proving triangulation independence.

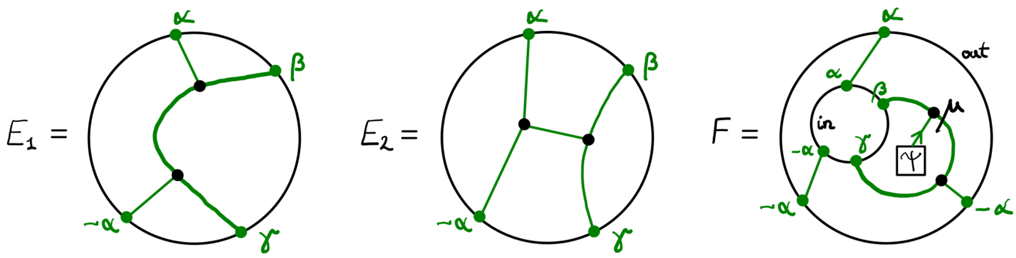
Lemma 4.44. Consider the morphisms $D_1, D_2 : \emptyset \to S^1$ in $\mathsf{Bord}_2^{\mathrm{def}}(S)$,



where $\alpha \in \{\pm\}$ and the inner edge without indicated orientation can be oriented any way such that the not yet labelled three-valent vertices can be labelled by either μ or Δ. Then (26.5.26)

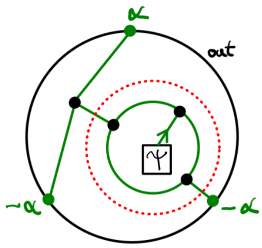
$$L(D_1) = L(D_2) \ .$$

Proof. Define $E_1, E_2 : \emptyset \to S^1$ and $F : S^1 \to S^1$ as follows:



Then $D_1 = [\emptyset \xrightarrow{E_1} S^1 \xrightarrow{F} S^1]$, and $L(D_1) = L(F) \circ L(E_1)$. We have "cut out the disc E_1". By Lemma 4.41, $L(E_1) = L(E_2)$, and so also $L(D_1) = L(F) \circ L(E_2) = L(F \circ E_2)$. This amounts to "gluing back the disc E_2". The result is

$$F \circ E_2 =$$



Now repeat this procedure by cutting out the disc indicated by the dotted circle and use Lemma 4.42. □

■ Recall that

$$Y = \left[\mathsf{Bord}_2^T \xrightarrow{D_{A,\mu,\Delta,\psi}} \mathsf{Bord}_2^{\mathrm{def}}(S) \xrightarrow{L} S\right] .$$

Proposition 4.45. Let $(M, T_M, O_M) : X \to Y$ be a morphism in Bord_2^T. Then $Y(M, T_M, O_M)$ is independent of T_M and O_M (for fixed (T_X, O_X) and (T_Y, O_Y)).

Proof. By Exercise 4.30 it is enough to show that $Y(M, T_M, O_M) = Y(M, T'_M, O'_M)$ if (T_M, O_M) and (T'_M, O'_M) differ by one oriented Pachner move. By Lemma 4.40 we may assume that the triangles affected by the Pachner move do not contain the 0-stratum labelled by ψ.

If (T_M, O_M) and (T'_M, O'_M) differ by a 2-2 move, cut out the PL-disc D_1 formed by these two triangles from the bordism $D(M, T_M, O_M)$ in $\mathsf{Bord}_2^{\mathrm{def}}(S)$, and denote the result by N, so that

$$D(M, T_M, O_M) = \left[X \xrightarrow{1_X \sqcup D_1} X \sqcup S^1 \xrightarrow{N} Y\right] .$$

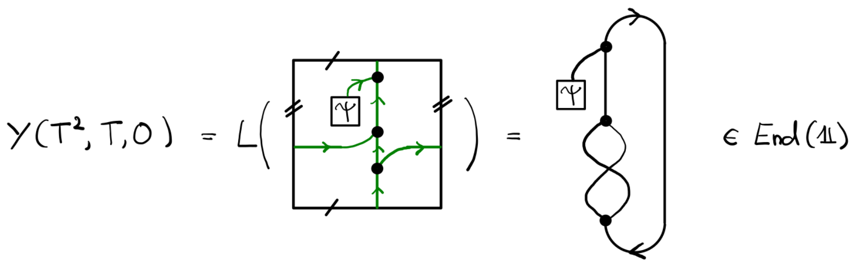
The disc D_1 is isomorphic to D_1 as in Lemma 4.41 for some choice of signs. The 2-2 Pachner move amounts to replacing D_1 by D_2 from Lemma 4.41:

$$D(M, T'_M, O'_M) = \left[X \xrightarrow{1_X \sqcup D_2} X \sqcup S^1 \xrightarrow{N} Y\right] .$$

Now apply the symmetric monoidal functor L to these two identities, and use that by Lemma 4.41 we have $L(D_1) = L(D_2)$. As $Y = L \circ D$, this shows $Y(M, T_M, O_M) = Y(M, T'_M, O'_M)$.

The argument for the 1-3 move is analogous, using now Lemma 4.44. □

Example 4.46. If M is a closed 2-manifold (a morphism $\emptyset \to \emptyset$), then $Y(M, T_M, O_M) \in \mathrm{End}(\mathbf{1})$ is independent of (T_M, O_M) and hence an invariant of M. For example, if $M = T^2$, a 2-torus, then a possible triangulation was shown below Definition 4.21. One can put a height order on this, compute $D(T^2, T, O)$, and then use Lemmas 4.40–4.42 to simplify the result without changing the value of L. Without going through the tedious details, the answer is



Let us compute this concretely in two examples and for $S = \mathsf{Vec}_k^{\mathrm{fd}}$.

1. For $\mathrm{Mat}_n(K)$, we have seen in Example 4.37 that $\psi = E_{11}$ is a possible choice of separability element. Using the coproduct

$$\Delta(M) = \sum_{i,j=1}^{n} (ME_{ij}) \otimes_K E_{ji} \ ,$$

we can evaluate the above diagram to (write E^*_{kl} for the dual basis),

$$\sum_{i,j,k,l=1}^{n} E^*_{kl} E_{11} E_{ji} E_{kl} E_{ij} = \sum_{i=1}^{n} E^*_{ii} E_{11} E_{1i} E_{i1} = 1 \ .$$

(Check that this is indeed the expression one obtains from evaluating the above diagram. Why does each step in the computation hold?) ?

2. For G a finite group such that $|G| \neq 0$ in k. In $k[G]$ the coproduct is $\Delta(b_g) = \sum_{g\in G} b_{gh} \otimes b_{h^{-1}}$ and we can take $\psi = |G|^{-1}$. The diagram evaluates to

$$|G|^{-1} \sum_{g,h\in G} b^*_g b_{h^{-1}} b_{gh} = |G|^{-1} \sum_{g,h\in G} \delta_{hg,gh} \ .$$

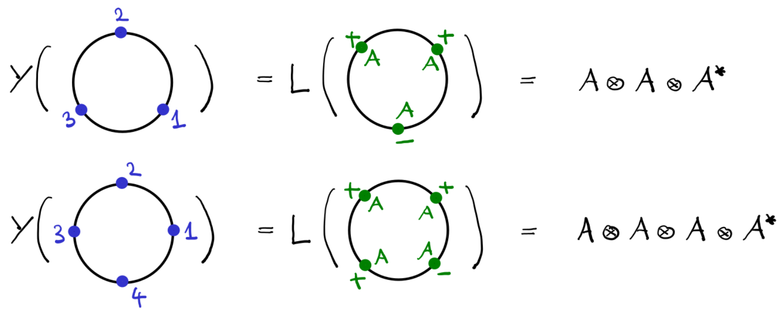
So the invariant is the number of pairs of commuting elements in G, divided by the group order $|G|$.

Combining Proposition 4.17 with Proposition 4.56 below, we will see that for $S = \mathsf{Vec}_k^{\mathrm{fd}}$, $Y(T^2, T, O)$ computes the dimension of the centre of A. In particular, it is necessarily a positive integer, which is not a priori obvious in the second example.

Exercise 4.47. Let A be a separable symmetric Frobenius algebra in a rigid symmetric monoidal category S. Show that for a sphere S^2, we have $Y(T^2, T, O) = \varepsilon \circ \psi$. Compute this number in the two examples treated above. E

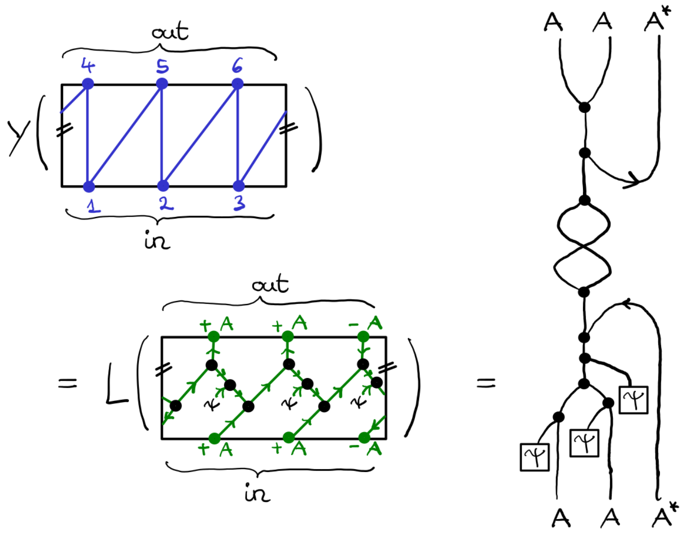
■ By Proposition 4.45, Y is independent of the triangulation of a morphism, but on objects it still depends on the triangulation, as the next example illustrates.

Example 4.48. 1. Here are two triangulations of S^1, the corresponding stratification and value of Y:



In general, these two objects in S are not isomorphic.

2. Let (S^1, T, O) be the triangulation in the first line in part 1, with 3 vertices. Let $(M, T_M, O_M) : (S^1, T, O) \to (S^1, T, O)$ be the cylinder over S^1, triangulated as shown below in the computation of $Y(M, T_M, O_M)$:



Write $\pi = Y(M, T_M, O_M)$. Then π is an idempotent: $\pi \circ \pi = \pi$. (In the last equality in the above diagram, some simplifications were done, can you reproduce the answer? Try to verify that the morphism in S on the right hand side of the above diagram is indeed an idempotent.) ?

■ The idempotent example above generalises: Let $X \in \mathsf{Bord}_2$ and set $X_j = (X, T_j, O_j)$, $j = 1, 2, 3$ for three choices of oriented triangulations (T_j, O_j) for X. Pick a triangulation (T, O) on $X \times I$ such that

$$(X \times I, T, O) : X_1 \to X_2 \ .$$

Then by Proposition 4.45,

$$\pi_{X_2,X_1} := Y(X \times I, T, O) : Y(X_1) \to Y(X_2)$$

is independent of the choice of (T, O), and we have

$$\pi_{X_3,X_1} := \pi_{X_3,X_2} \circ \pi_{X_2,X_1} : Y(X_1) \to Y(X_3) \ .$$

This identity is a special case of the next lemma. (Why?) For later use we note that in particular, π_{X_1,X_1} is an idempotent. ?

Lemma 4.49. Let $M : W \to X$ be a morphism in Bord_2. Let $(T_{W,j}, O_{W,j})$, $(T_{X,j}, O_{X,j})$, $j = 1, 2$ be oriented triangulations of X and W. Let (T_{ij}, O_{ij}) be oriented triangulations of M so that we get bordisms

$$(M, T_{ij}, O_{ij}) : (W, T_{W,j}, O_{W,j}) \to (X, T_{X,i}, O_{X,i})$$

in Bord_2^T. Then

$$Y(M, T_{11}, O_{11}) \;=\; Y(M, T_{12}, O_{12}) \circ \pi_{W_2,W_1} \;=\; \pi_{X_1,X_2} \circ Y(M, T_{21}, O_{21}) \ .$$

Proof. We show the first identity, the second one can be checked analogously. We compute

$$\begin{aligned} Y(M, T_{12}, O_{12}) \circ \pi_{W_2,W_1} &= Y(M, T_{12}, O_{12}) \circ Y(W \times I, T, O) \\ &\overset{(1)}{=} Y\big((M, T_{12}, O_{12}) \circ (W \times I, T, O) \big) \\ &\overset{(2)}{=} Y(M, T'_{11}, O'_{11}) \overset{(3)}{=} Y(M, T_{11}, O_{11}) \end{aligned}$$

In step (1) we use functoriality of Y, step (2) is the observation that gluing a cylinder over W to the incoming boundary W of M is isomorphic to M, but with a different oriented triangulation (T'_{11}, O'_{11}). Step (3) is Proposition 4.45 that Y is independent of the triangulation. □

For $X_1 = (X, T_1, O_1)$ abbreviate $\pi_{X_1} = \pi_{X_1,X_1}$. Recall that π_{X_1} is an idempotent.

Corollary 4.50. $\pi = (\pi_{(X,T_X,O_X)})$ is a natural endomorphism of Y.

?

(Why is this immediate from Lemma 4.49?)

■ Suppose that S is idempotent complete. Consider the category $C = \mathrm{Fun}(\mathsf{Bord}_2^T, S)$ of functors and natural transformations. (Why is C a category and not just a semicategory?) Then $Y \in C$ and $\pi : Y \Rightarrow Y$ is an ?

idempotent. We will construct a splitting $(\tilde{Z}, \varepsilon, \rho)$ of Y in C, that is, a functor $\tilde{Z} : \mathsf{Bord}_2^T \to S$ and natural transformations

$$\varepsilon : \tilde{Z} \Rightarrow Y \ , \quad \rho : Y \Rightarrow \tilde{Z} \ , \quad \varepsilon \circ \rho = \pi \ , \quad \rho \circ \varepsilon = \mathrm{id} \ .$$

Moreover, $\tilde{Z}$ factors as $\tilde{Z} = Z \circ U$ as in the diagram we were aiming for:

$$\begin{tikzcd} \mathsf{Bord}_2^T \arrow[r, "Y"] \arrow[d, "U"'] & S \\ \mathsf{Bord}_2 \arrow[ur, dashed, "Z"'] & \end{tikzcd}$$

Even though the construction of Z, $\tilde{Z}$, ε, ρ below will depend on additional choices, the fact that it can be expressed as the splitting of an idempotent will make Z unique up to isomorphism.

■ For each $X \in \mathsf{Bord}_2$ pick an oriented triangulation (T_X^0, O_X^0). Pick a retract (j, e, r) of $\pi_{(X, T_X^0, O_X^0)}$ and define

$$\begin{aligned} Z(X) = j \ , \quad \varepsilon_{(X,T_X^0,O_X^0)} &= e : Z(X) \to Y(X, T_X^0, O_X^0) \ , \\ \rho_{(X,T_X^0,O_X^0)} &= r : Y(X, T_X^0, O_X^0) \to Z(X) \ . \end{aligned}$$

Using these, we can define Z on morphisms. Let $M : X \to W$ be a bordism in Bord_2. Pick a triangulation (T, O) of M such that

$$(M, T, O) : (X, T_X^0, O_X^0) \to (W, T_W^0, O_W^0) \ .$$

We set

$$Z(M) = \Big[Z(X) \xrightarrow{\varepsilon_{(X,T_X^0,O_X^0)}} Y(X, T_X^0, O_X^0) \xrightarrow{Y(M,T,O)} Y(W, T_W^0, O_W^0) \xrightarrow{\rho_{(W,T_W^0,O_W^0)}} Z(W) \Big]$$

Lemma 4.51. ε, ρ as above extend uniquely to natural transformations (27.5.26)

$$\varepsilon : Z \circ U \Rightarrow Y \ , \quad \rho : Y \Rightarrow Z \circ U \quad \text{s.t.} \quad \varepsilon \circ \rho = \pi \ , \quad \rho \circ \varepsilon = \mathrm{id} \ .$$

Proof. We start by making an ansatz for the natural transformations ε, ρ. Let (T_1, O_1) be an arbitrary triangulation of X. Abbreviate $X^0 = (X, T_X^0, O_X^0)$ and $X_1 = (X, T_1, O_1)$. Define the ε_{X_1} and ρ_{X_1} as

$$\begin{tikzcd} Z(X) \arrow[r, "\varepsilon_{X^0}"] \arrow[dr, dashed, "\varepsilon_{X_1}"'] & Y(X^0) \arrow[d, "\pi_{X_1,X^0}"] \\ & Y(X_1) \end{tikzcd} \quad , \quad \begin{tikzcd} Y(X^0) \arrow[r, "\rho_{X^0}"] & Z(X) \\ Y(X_1) \arrow[u, "\pi_{X^0,X_1}"] \arrow[ur, dashed, "\rho_{X_1}"'] & \end{tikzcd} \ .$$

For $X_1 = X^0$, this reads $\varepsilon_{X^0} = \pi_{X^0,X^0} \circ \varepsilon_{X^0}$ and $\rho_{X^0} = \rho_{X^0} \circ \pi_{X^0,X^0}$. Both identities hold for idempotent splittings, cf. Exercise 3.19 (1). Hence the notation is consistent.

Now we check the required properties:

- $\rho \circ \varepsilon = \mathrm{id}$: We compute on the object X_1,

$$\begin{aligned} \rho_{X_1} \circ \varepsilon_{X_1} &= \rho_{X^0} \circ \pi_{X^0,X_1} \circ \pi_{X_1,X^0} \circ \varepsilon_{X^0} = \rho_{X^0} \circ \pi_{X^0,X^0} \circ \varepsilon_{X^0} \\ &= \rho_{X^0} \circ \varepsilon_{X^0} = 1_{ZU(X_1)} . \end{aligned}$$

  The first step is just the definition, the second step uses the identity above Lemma 4.49, the third step is Exercise 3.19 (1), and the last step is the fact that $(Z(X), \varepsilon_{X^0}, \rho_{X^0})$ is by definition a splitting of the idempotent π_{X^0,X^0}.

- $\varepsilon \circ \rho = \pi$: shown similarly. (Details?) ?

- *Naturality of* ε, ρ: We will show naturality for ε, that of ρ can be seen similarly. Let $M : X \to W$ be a morphism in Bord_2, and let $M_1 : X_1 \to W_1$ be a triangulation thereof, that is, a bordism in Bord_2^T that gets mapped to M by the forgetful functor: $U(M_1) = M$. Similarly, pick $M^0 : X^0 \to W^0$ with $U(M^0) = M$. The outer square of the following diagram is the naturality square:

$$\begin{tikzcd}
ZU(X_1) \arrow[rrrr, "ZU(M_1)"] \arrow[dd, "\varepsilon_{X_1}"'] \arrow[dr, "\varepsilon_{X^0}"] & & (2) & & ZU(W_1) \arrow[dd, "\varepsilon_{W_1}"] \arrow[dl, "\varepsilon_{W^0}"'] \\
(3) & Y(X^0) \arrow[rr, "Y(M^0)"] \arrow[dl, "\pi_{X_1,X^0}"] & & Y(W^0) \arrow[dr, "\pi_{W_1,W^0}"] & (1) \\
Y(X_1) \arrow[rrrr, "Y(M_1)"'] & & (4) & & Y(W_1)
\end{tikzcd}$$

  The triangles and square (1), (2), (3) of the diagram commute by definition. Square (4) is naturality of π in Corollary 4.50.

- *Uniqueness of* ε, ρ: The above naturality square, applied to a triangulation $M_1 : X^0 \to X_1$ of the identity morphism $M = X \times I : X \to X$ in Bord_2, gives

$$\begin{tikzcd}
Z(X) \arrow[r, "Z(X\times I)=1_{Z(X)}"] \arrow[d, "\varepsilon_{X^0}"] & Z(X) \arrow[d, "\varepsilon_{X_1}"] \\
Y(X^0) \arrow[r, "Y(M_1)=\pi_{X_1,X^0}"] & Y(X_1)
\end{tikzcd}$$

Thus $\varepsilon_{X_1} = \pi_{X_1,X^0} \circ \varepsilon_{X^0}$ is forced by naturality. The argument for ρ is analogous.

$\square$

■ Finally, we need to make Z symmetric monoidal. For this, first note that (with appropriate ordering conventions), for $M : X \to W$ and $M' : X' \to W'$ in Bord_2^T,

$$D(\emptyset) = \emptyset \ , \quad D(X \sqcup X') = D(X) \sqcup D(X') \ , \quad D(M \sqcup M') = D(M) \sqcup D(M') \ .$$

Furthermore, $L : \mathsf{Bord}_2^{\mathrm{def}}(S) \to S$ is symmetric monoidal, and so we get isomorphisms, natural in X, X',

$$\underbrace{L\big(D(X) \sqcup D(X')\big)}_{=Y(X \sqcup X')} \xrightarrow{L^2_{D(X),D(X')}} \underbrace{L(D(X)) \otimes L(D(X'))}_{=Y(X) \otimes Y(X')} \ .$$

Lemma 4.52. Let Z, ε, ρ be a splitting of π as in Lemma 4.51. Then there exists a unique symmetric monoidal structure on Z such that

$$\begin{tikzcd}
Y(X \sqcup X') \arrow[r, "L^2_{D(X),D(X')}"] & Y(X) \otimes Y(X') \arrow[d, "\rho_X \otimes \rho_{X'}"] \\
ZU(X \sqcup X') \arrow[u, "\varepsilon_{X \sqcup X'}"] \arrow[r, dashed, "Z^2_{U(X),U(X')}"] & ZU(X) \otimes ZU(X')
\end{tikzcd}$$

commutes for all $X, X' \in \mathsf{Bord}_2^T$.

Proof. Uniqueness is clear as the diagram determines Z^2. Existence requires that $Z^2_{U(X),U(X')}$ is well-defined, i.e. does not depend on the triangulations chosen on X, X'. To see this, choose another triangulation $\tilde{X}$ of the same underlying 1-manifold, $U(\tilde{X}) = U(X)$, and analogously for $\tilde{X}'$. Commutativity of the following diagram can then be used to show $Z^2_{U(X),U(X')} = Z^2_{U(\tilde{X}),U(\tilde{X}')}$:

$$\begin{tikzcd}
Y(X \sqcup X') \arrow[r, "L^2_{D(X),D(X')}"] \arrow[d, "\pi_{\tilde{X} \sqcup \tilde{X}', X \sqcup X'}"'] & Y(X) \otimes Y(X') \arrow[d, "\pi_{\tilde{X},X} \otimes \pi_{\tilde{X}',X'}"] \\
Y(\tilde{X} \sqcup \tilde{X}') \arrow[r, "L^2_{D(\tilde{X}),D(\tilde{X}')}"] & Y(\tilde{X}) \otimes Y(\tilde{X}')
\end{tikzcd}$$

The diagram commutes by naturality of L^2. (How does this imply that the two expressions for Z^2 agree?) ? Symmetry of Z follows from that of L. $\square$

For info: The fact that Z (or rather $Z \circ U$) arises via an idempotent splitting shows that it is unique up to natural isomorphism. From this one can show two further universal properties: Z is both a left and a right Kan extension of the diagram

$$\begin{tikzcd} \mathsf{Bord}_2^T \arrow[r, "Y"] \arrow[d, "U"'] & S \\ \mathsf{Bord}_2 \arrow[ur, dashed, "Z"'] & \end{tikzcd}$$

See [Ri, Ch. 4] for more on Kan extensions.

Let us collect the construction in a theorem.

Theorem 4.53. Let S be an idempotent complete rigid symmetric monoidal category, and A a separable symmetric Frobenius algebra. The there is a unique (up to unique natural monoidal isomorphism) $Z_A \in \mathsf{TFT}_2(S)$ together with natural transformations ε, ρ as in Lemma 4.51 and monoidal structure determined by Lemma 4.52.

Proof. Existence follows form the explicit construction above Lemma 4.51, and uniqueness-up-to-unique-isomorphism follows from Lemma 4.51 and the uniqueness of splittings of idempotents, cf. Exercise 3.19 (1). The symmetric monoidal structure was constructed in Lemma 4.52. With respect to this monoidal structure, one checks that the unique natural isomorphism between two splittings is monoidal. □

For info: One nice application of 2d state sum TFTs to group theory is Mednykh's formula: Let G be a finite group and M a closed, oriented, connected 2-manifold. Then

$$|G|^{\chi(M)-1} \left| \mathrm{Hom}(\pi_1(M), G) \right| = \sum_{V \in \hat{G}} (\dim V)^{\chi(M)} .$$

Here, $\pi_1(M)$ is the fundamental group of M, and $\mathrm{Hom}(\dots)$ stands for the set of group homomorphisms. The sum is over $\hat{G}$, the isomorphism classes of irreducible representations of G. The TFT-based proof starts from the algebra isomorphism

$$\mathbb{C}[G] \cong \bigoplus_{V \in \hat{G}} \mathrm{End}_{\mathbb{C}}(V)$$

which write the semisimple algebra $\mathbb{C}[G]$ as a direct sum of matrix algebras $\mathrm{End}_{\mathbb{C}}(V)$. This can be turned into an isomorphism of separable symmetric Frobenius algebras. Hence the 2d TFT for both algebras will assign the same value to M, and this produces the identity above. See [Sn] for details of the TFT proof and a historical account of Mednykh's Formula.

Exercise 4.54. Let $(A, \mu, \eta, \Delta, \varepsilon, \psi)$ be a separable symmetric Frobenius algebra in Vec_k and $\alpha, \beta \in k^\times$. (More generally, one can use a symmetric monoidal category S and invertible $\alpha, \beta \in \mathrm{End}_S(\mathbb{1})$, but let us stick to vector spaces here.) Ⓔ

1. Show that A with $\alpha\eta$, $\alpha^{-1}\mu$, $\beta\varepsilon$, $\beta^{-1}\Delta$, $\alpha^2\beta\psi$ is a separable symmetric Frobenius algebra. Denote this algebra by $A_{\alpha,\beta}$. Show that for $\alpha\beta = \gamma\delta$ we have that $A_{\alpha,\beta} \cong A_{\gamma,\delta}$ as separable Frobenius algebra, that is, the Frobenius algebra isomorphism also has to preserve the separability element.

2. Write $B = A_{\alpha,1}$. Let M be a closed, 2-manifold with oriented triangulation, that is, a bordism $M : \emptyset \to \emptyset$ in Bord_2^T. Show

   $$Y_B(M) = \alpha^{\chi(M)}\, Y_A(M) \ ,$$

   with χ the Euler characteristic of M.

   Hint: Consider $Y_A(M, T, O)$. On (M, T, O), assign a factor of α to each vertex of T, a factor of α^{-1} to each edge and a factor of α to each triangle. Pass to the stratification on M produced by $D(M, T, O)$. "Push" the α^{-1}-factor along the oriented A-labelled 1-stratum into the 0-stratum (labelled μ or Δ) at the end of the 1-stratum. Note that each ψ comes with its own μ giving an overall α.

3. For a bordism $M : X \to Y$ in Bord_2^T show

   $$\alpha^{N_-(Y) - N_-(X)}\, Y_B(M) = \alpha^{\chi_b(M)} Y_A(M) \ .$$

   Here, $N_-(X)$ is the number of "-"-oriented 0-strata on $D(X)$, and χ_b is the bordism Euler characteristic defined above Proposition 4.22. However, in the case of 2d bordisms, we actually have $\chi_b(M) = \chi(M)$ as the Euler characteristic of S^1 is zero.

   Hint: Proceed as in part 2 and count the left-over factors. Recall from the construction of D that there are no 0-strata labelled by ψ associated to vertices contained in Y.

4. Use part 3 and the definition of Z_A to show that

   $$Z_B \cong Z_A \otimes E_\alpha \ ,$$

   where E_α is the invertible TFT from Proposition 4.22.

   Hint: Fix a standard triangulation (T_S, O_S) on S^1 so that $D(S^1, T_X, O_S)$ contains two 0-strata $\bullet_+^A$ and one 0-stratum $\bullet_-^A$. Then $N_-(S^1) = 1$ and hence $N_-(X) = |X|$, the number of connected components of X, that is, the number of disjoint copies of S^1 making up X. Note that Y_A and Y_B have the same values on cylinders, i.e. give the same idempotents, and so the state spaces of Z_A and Z_B are the same. Try $\phi_X = \alpha^{|X|}\,\mathrm{id}$ as natural monoidal isomorphism $Z_B \Rightarrow Z_A \otimes E_\alpha$. Here, id is understood as a linear map $Z_A(X) \to Z_B(X)$.

For info: Part 3 shows that 2d state sum TFTs close under stacking with invertible TFTs – this would not be true if we had restricted ourselves to the (simpler) case $\psi = \eta$. A separable Frobenius algebra with $\psi = \eta$ is sometimes called Δ-separable, because the element D in Exercise 4.39 is in this case just the coproduct Δ.

■ In Theorem 4.14 we have seen that the groupoid $\mathsf{TFT}_2(S)$ is equivalent to $\mathsf{ComFrob}(S)$. So Z_A in Theorem 4.53 corresponds to some commutative Frobenius algebra $C \in S$. We will describe C in the case $S = \mathsf{Vec}_k$. We need:

Exercise 4.55. Let A be an algebra in Vec_k. The *centre of* A is Ⓔ

$$C(A) = \{c \in A \,|\, ac = ca \text{ for all } a \in A\} \ .$$

Let now $A = (A, \mu, \eta, \Delta, \varepsilon, \psi)$ be a separable symmetric Frobenius algebra in Vec_k. Define

$$\tau : A \to A \ , \quad \tau = \mu \circ \beta_{A,A} \circ \Delta \quad , \qquad p : A \to A \ , \quad p(a) = \tau(\psi a) \ .$$

Lemma 4.34 is useful in the problems below.

1. Show that p is an idempotent whose image is $C(A)$. *Hint:* Use string diagrams. Show $C(A) \subset \mathrm{im}(p)$ and $\mathrm{im}(p) \subset C(A)$.

2. Define $\gamma_C = (\tau \otimes p) \circ \Delta \circ \eta$. Show that $\gamma_C = (p \otimes \tau) \circ \Delta \circ \eta$ and that

$$\gamma_C = (\mathrm{id}_A \otimes \tau) \circ \Delta \circ \eta.$$

   Show that $(p \otimes p) \circ \gamma_C = \gamma_C$ and conclude that for $\gamma_C \in C(A) \otimes C(A)$.

3. Consider the pairing $C(A) \times C(A) \to k$, $(a, b) \mapsto \varepsilon(\psi ab)$. Show that this pairing is non-degenerate. *Hint:* Try γ_C as copairing.

4. Show that $C(A)$ with the same algebra structure as A and coalgebra structure, for $c \in C(A)$,

$$\Delta_C(c) = (\tau \otimes p) \circ \Delta(c) \ , \quad \varepsilon_C(c) = \varepsilon(\psi c)$$

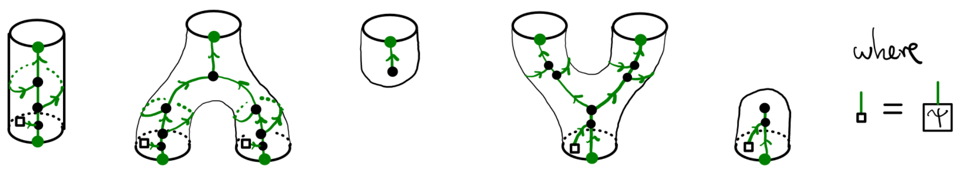
   is a commutative Frobenius algebra (which is then also symmetric). *Hint:* Check that Δ_C is of the form given in Exercise 4.7.

We can now state:

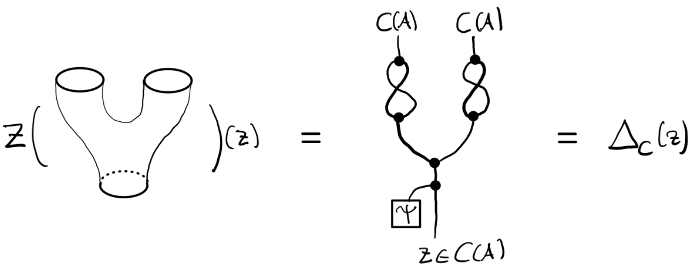
Proposition 4.56. Let A be a separable symmetric Frobenius algebra in $\mathsf{Vec}_k^{\mathrm{fd}}$. The equivalence in Theorem 4.14 maps the TFT Z_A from Theorem 4.53 to the commutative Frobenius algebra $C(A)$, the centre of A.

Sketch of proof. We will only give the outline of the argument. The details are not difficult, but a bit tedious.

As triangulations are too cumbersome for this argument, one first shows that rather than the stratifications produced by $D : \mathsf{Bord}_2^T \to \mathsf{Bord}_2^{\mathrm{def}}$, one can use stratifications in $\mathsf{Bord}_2^{\mathrm{def}}$ with the property that all 1-cells and 2-cells are contractible (that is, isomorphic to intervals or discs). Then the stratifications for cylinder, and for the generating morphisms in Theorem 4.14 are



(Can you show that $p = L(\text{cylinder}) : A \to A$ as above has an image isomorphic to that of the idempotent in Example 4.48 (2), where we used an actual triangulation of the cylinder?) ? One now checks that this produces the commutative Frobenius algebra structure from Exercise 4.55. For example, for the coproduct we have, for $z \in C(A) = Z(S^1)$:



□

For info: 1. Can define the notion of a centre of an algebra in general symmetric monoidal categories. Existence is in general not guaranteed, but the centre does exist if S is abelian (has kernels and cokernels, satisfying an extra condition). For separable Frobenius algebras, it is enough that S is idempotent complete for the centre to exist. The statement in Proposition 4.56 holds in general for idempotent complete symmetric monoidal categories S.

2. If S is just braided and not symmetric, there are two distinct notions of a centre, the left and right centre [FFRS, Sec. 2.4]. One can even define the centre of an algebra in just a monoidal category. The centre in that case is a commutative algebra in the Drinfeld centre of the original category, see [Da].

3. Proposition 4.56 also raises the question if all 2d TFTs arise for a state sum construction. The answer to this is "no", at least naively in the sense that state sum TFTs constructed from separable symmetric Frobenius algebras in S as in Theorem 4.53 need not give all 2d TFTs obtained from commutative Frobenius algebras in S via Theorem 4.14. An example is $S = \mathsf{Vec}_{\mathbb{C}}$, together with the observation that a separable algebra is semisimple and hence a direct sum of full matrix algebras. Its centre is the sum of copies of $\mathbb{C}$ and is again semisimple. Hence Example 4.8 (4) can not be obtained from such a state sum construction. However, such algebras can be obtained form more elaborate state-sum-like constructions, for example in the symmetric monoidal bicategory of potentials and matrix factorisations [CMM].

For info: Outlook: We will now continue with three-dimensional TFT. Some natural topics in which one could continue in two dimensions are:

1. The state sum construction above can be extended to produce examples of 2d defect TFTs. The 2-strata are be labelled by separable symmetric Frobenius algebras, and the 1-strata by bimodules over the neighbouring algebras. See [DKR, Sec. 3].

2. Just as state sums without defects could reproduce interesting lattice models, such as the Ising lattice model, state sums constructions with defects can produce lattice models with line defects. It can happen that the line defects can still move across the lattice, giving a non-topological lattice model with topological line defects. See e.g. [CDR, Sec. 2] and [AFM].

3. A 2d defect TFT with a unique label for 2-strata gives rise to a pivotal monoidal category, or, if there is more than one possible label for 2-strata, to a bicategory with twosided adjoints. See [DKR, Sec. 2.4].

4. For a given 2d defect TFT Z with a unique label for 2-strata, let $\mathcal{E}$ "endo-defects" be the pivotal monoidal category from part 2. Given a pivotal monoidal category C and a pivotal monoidal functor $C \to \mathcal{E}$, we say "Z has categorical symmetry C". One can ask, what the smallest 2d TFT (in the sense of having $Z(S^1)$ of lowest dimension) is that can have a given categorical symmetry $\mathcal{C}$. For example, if $Z(S^1)$ is one-dimensional, and we ask $\mathcal{C}$ to in addition be a fusion category (more on those in the next section), then the TFT Z defines a fibre functor $\mathcal{C} \to \mathsf{Vec}_k$. By reconstruction, this means that $\mathcal{C} \cong \mathrm{Rep}(H)$ for some semisimple Hopf algebra H. Not all fusion categories admit fibre functors. See e.g. [CDR, Sec. 3.3]

5. The previous point has an application in Physics. Namely, if a 2d conformal field theory is perturbed in a way that preserves a categorical symmetry $\mathcal{C}$, then one expects the renormalisation group flow end point to still carry categorical symmetry $\mathcal{C}$. If the endpoint is a massive theory, its ground states should form a 2d TFT with categorical symmetry $\mathcal{C}$. If $\mathcal{C}$ cannot be realised via a 2d TFT with dim $Z(S^1) = 1$, the flow endpoint cannot have a unique ground state: "The vacuum state of the infrared QFT spontaneously breaks the $\mathcal{C}$-symmetry."

(3.6.26)

Bibliographic notes for Section 4

Frobenius algebras in monoidal categories are described for example in [FSt] and [Ko, Sec. 3.6]. The equivalence between the two notions of separability in Exercise 4.39 is due to [Mul, Prop. 2.10].

The earliest reference for the equivalence between 2d TFTs and commutative Frobenius algebras I am aware of is [Dij, Sec. 3.2]. The equivalence of groupoids of 2d TFTs and commutative Frobenius algebras (in vector spaces) is shown in [Ab]. A textbook reference is [Ko].

For a more general description of invertible TFTs, see e.g. [SP] and the lecture notes [Fr, Sec. 6.2]. Euler TFTs are also given in [Qu, Lect. 3], and, for stratified bordisms, in [CRS2, Ex. 2.14].

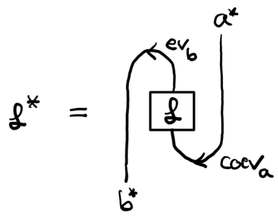
The trivial n-dimensional TFT with line defects (with S being super-vector spaces) is constructed in 3d in [RSW, Sec. 4.2], and in general in [CN, Sec. 7].

State sum constructions of topological field theories have first been given in two dimensions in [BP, FHK]. A version for separable algebras has been described in [LP] (with the assumption that $\psi : \mathbb{1} \to A$ is central), and in [Mul, Sec. 4] in general. The particular approach to state sums by mapping them to the trivial TFT with line defects seems to be new. The same holds for the description of non-topological state sum models (in the example of the Ising model) via the trivial TFT with defects. The closest works for the latter are maybe the construction of 2d lattice models via fusion categories in [AFM] or of tensor networks on the boundary of 3d TFT in [LFHSV].

5 Dimension three

5.1 Ribbon categories

■ Before stating the definition, we need one more notion from monoidal categories: Let $\mathcal{C}$ be a monoidal category, suppose $a, b \in \mathcal{C}$ have left duals a^*, b^*, and let $f : a \to b$ be given. Then the *(left) dual morphism* $f^* : b^* \to a^*$ is defined as



(Write out f^* with all coherence isomorphisms included.) Equivalently, f^* is uniquely determined by any one of the following two conditions: ?

$$\begin{aligned} \mathrm{ev}_b \circ (1_{b^*} \otimes f) &= \mathrm{ev}_a \circ (f^* \otimes 1_a) && : \ b^* \otimes a \to \mathbb{1} \ , \\ (f \otimes 1_{a^*}) \circ \mathrm{coev}_a &= (1_b \otimes f^*) \circ \mathrm{coev}_b && : \ \mathbb{1} \to b \otimes a^* \ . \end{aligned}$$

(Why are these three definitions of f^* equivalent?) If a, b have right duals, one can analogously define the right dual of f. ?

Definition 5.1. Let $\mathcal{C}$ be a braided monoidal category.

1. $\mathcal{C}$ is called *balanced* if it is equipped with a natural isomorphism θ of the identity functor on $\mathcal{C}$, called *balancing isomorphism*, whose components $\theta_U : a \to a$ satisfy, for all $a, b \in \mathcal{C}$

$$\theta_{a \otimes b} = (\theta_a \otimes \theta_b) \circ \beta_{b,a} \circ \beta_{a,b} \ : \ a \otimes b \to a \otimes b \ .$$

2. $\mathcal{C}$ is called *ribbon* if it has left duals, is balanced, and the balancing isomorphism satisfies, for all $a \in \mathcal{C}$,

$$\theta_{a^*} = (\theta_a)^* \ : \ a^* \to a^* \ .$$

In a ribbon category, θ is also called the *ribbon twist*.

Let $\mathcal{C}, \mathcal{D}$ be balanced (resp. ribbon) categories. A *balanced* (resp. *ribbon*) *functor* is a braided monoidal functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(\theta_a) = \theta_{F(a)} : F(a) \to F(a)$, for all $a \in \mathcal{C}$.

Remark 5.2. 1. The balancing isomorphisms automatically satisfy $\theta_{\mathbb{1}} = 1_{\mathbb{1}}$. Indeed, for $a = b = \mathbb{1}$, the defining condition reads $\theta_{\mathbb{1}\otimes\mathbb{1}} = \theta_{\mathbb{1}} \otimes \theta_{\mathbb{1}}$. Composing both sides with the left unitor of $\mathcal{C}$ gives

$$\lambda_{\mathbb{1}} \circ \theta_{\mathbb{1}\otimes\mathbb{1}} \overset{(1)}{=} \theta_{\mathbb{1}} \circ \lambda_{\mathbb{1}} \ ,$$

$$\begin{aligned}\lambda_{\mathbb{1}} \circ (\theta_{\mathbb{1}} \otimes \theta_{\mathbb{1}}) &\overset{(2)}{=} \theta_{\mathbb{1}} \circ \lambda_{\mathbb{1}} \circ (\theta_{\mathbb{1}} \otimes 1_{\mathbb{1}}) \overset{(3)}{=} \theta_{\mathbb{1}} \circ \rho_{\mathbb{1}} \circ (\theta_{\mathbb{1}} \otimes 1_{\mathbb{1}}) \\ &\overset{(4)}{=} \theta_{\mathbb{1}} \circ \theta_{\mathbb{1}} \circ \rho_{\mathbb{1}} \overset{(3)}{=} \theta_{\mathbb{1}} \circ \theta_{\mathbb{1}} \circ \lambda_{\mathbb{1}} \ ,\end{aligned}$$

where: (1) naturality of θ; (2) naturality of λ; (3) $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$; (4) naturality of ρ. As all maps involved are isomorphisms, this shows $\theta_{\mathbb{1}} = 1_{\mathbb{1}}$.

2. A symmetric monoidal category S is balanced via $\theta_a = 1_a$ for all $a \in S$. Indeed, in this case the defining condition reads $1_{a\otimes b} = \beta_{b,a} \circ \beta_{a,b}$, which is the condition for the braiding β to be symmetric. (But S may allow for other balancing isomorphisms as well, see the lemma below.) Accordingly, S^{fd} is ribbon.

   Conversely, a balanced category with $\theta_a = 1_a$ for all a is necessarily symmetric. (Why?) ?

3. Let $\mathcal{C}$ be a balanced category with left duals. Looking at the equivalent definitions of a dual morphism, we see that $\mathcal{C}$ is ribbon if and only if, for all $a \in \mathcal{C}$,

$$\mathrm{ev}_a \circ (1_{a^*} \otimes \theta_a) = \mathrm{ev}_a \circ (\theta_{a^*} \otimes 1_a) \ : \ a^* \otimes a \to \mathbb{1} \ .$$

   An analogous equivalent condition exists for the coevaluation.

4. There is actually no difference betweeen a balanced and a ribbon functor, we could equally well have called both a balanced functor. But we would like to speak of ribbon functors between ribbon categories.

■ Any two balancing isomorphisms differ by a natural monoidal isomorphism of the identity functor. This is helpful in understanding all balancing structures on a braided category, provided one can find at least one.

Lemma 5.3. Let $\mathcal{C}$ be a braided monoidal category, and let θ be a balancing isomorphism on $\mathcal{C}$. There is a bijection

$$\{\text{nat. mon. iso. of } \mathrm{Id}_{\mathcal{C}}\} \to \{\text{balancing isomorphisms on } \mathcal{C}\}$$

given by $\eta \mapsto \eta \circ \theta$.

Exercise 5.4. Prove the above lemma. *Hint:* Show that $\eta \circ \theta$ is a balancing isomorphism, and that given another balancing isomorphism θ', the composition $\theta' \circ \theta^{-1}$ is a natural monoidal isomorphism of the identity. E

Example 5.5. Let G be an abelian group and $q : G \to k^\times$ a quadratic form on G. Pick an abelian 3-cocycle (ω, ψ) with $\psi(g, g) = q(g)$ (this determines $[\omega, \psi] \in H^3_{\mathrm{ab}}(G, k^\times)$ uniquely by Theorem 2.43). Consider the braided monoidal category $\mathcal{C} = \mathsf{Vec}_G^{(\omega,\psi)}$ as in and above Exercise 2.41.

1. For a homogeneous element u in $V = \bigoplus_{g\in G} V_g$ we set

$$\theta_V(u) = q(|u|)\, u \ .$$

   Then θ is a balancing isomorphism for $\mathcal{C}$. This will be shown in the next exercise.

2. Let $\chi : G \to k^\times$ be a group homomorphism, that is, a character of G. For $V \in \mathcal{C}$ and a homogeneous element $u \in V$ set

$$\eta_V(u) = \chi(|u|)\, u \ .$$

   Then η is a natural monoidal isomorphism of the identity functor on $\mathcal{C}$, and in fact all natural monoidal isomorphisms are of this form. (Why?) ?
   Accordingly, by Lemma 5.3, all balancing isomorphism on $\mathcal{C}$ are given by

$$\theta'_V(u) = \chi(|u|)\, q(|u|)\, u \ .$$

   Let us denote the balanced category with balancing isomorphism θ' as above by

$$\mathsf{Vec}_G^{(\omega,\psi,\chi)} \ .$$

   For example for $k = \mathbb{C}$, as G is abelian, it has $|G|$ many distinct characters, and hence there are $|G|$ distinct ways to turn $\mathsf{Vec}_G^{(\omega,\psi)}$ into a balanced category.

   Below we will write θ instead of θ' for the balancing isomorphism of $\mathsf{Vec}_G^{(\omega,\psi,\chi)}$.

3. Consider now $\mathcal{C}^{\mathrm{fd}} = \big(\mathsf{Vec}_G^{(\omega,\psi,\chi)}\big)^{\mathrm{fd}}$, consisting of finite-dimensional graded vector spaces. We would like to know for which choice of χ, if any, $\mathcal{C}$ is a ribbon category. The left dual of $V \in \mathcal{C}$ is

$$V^* = \bigoplus_{g\in G} (V^*)_g \quad \text{with} \quad (V^*)_g = (V_{g^{-1}})^* \ ,$$

that is the degree-g-component of V^* is the dual of $V_{g^{-1}}$. (What would go wrong if we would instead take $(V^*)_g = (V_g)^*$?) The evaluation map is the same as that of vector spaces

?

$$\mathrm{ev}_V : V^* \otimes V \to k \quad , \quad \varphi \otimes v \mapsto \varphi(v) \ ,$$

The coevaluation map, however, is not, see again the exercise below. The condition in Remark 5.2 (3) for a balancing isomorphism to define a ribbon twist becomes, for $u \in V$ homogeneous of degree g and $\varphi \in V^*$ of degree h,

$$\begin{aligned}\big(\mathrm{ev}_V \circ (\mathrm{id}_{V^*} \otimes \theta_V)\big)(\varphi \otimes u) &= \varphi(\theta_V(u)) = \chi(g)\, q(g)\, \varphi(u) \ , \\ \big(\mathrm{ev}_V \circ (\theta_{V^*} \otimes \mathrm{id}_V)\big)(\varphi \otimes u) &= (\theta_{V^*}(\varphi))(u) = \chi(h)\, q(h)\, \varphi(u) \ .\end{aligned}$$

By construction, $\varphi(u)$ is non-zero only for $h = g^{-1}$, and we conclude that $\mathcal{C}$ is ribbon if and only if

$$\chi(g)\, q(g) = \chi(g^{-1})\, q(g^{-1}) \ .$$

By definition, $q(g) = q(g^{-1})$, and we get a condition on χ:

$$\chi(g) \in \{\pm 1\} \quad \text{for all} \quad g \in G \ .$$

If k is not of characteristic 2, this means that the different ribbon structures on $\big(\mathsf{Vec}_G^{(\omega,\psi)}\big)^{\mathrm{fd}}$ are parametrised by group homomorphisms $\chi : G \to \{\pm 1\}$.

Exercise 5.6. In the notation of Example 5.5: E

1. Show that the bicharacter $b(g,h) = q(gh)/(q(g)q(h))$ (Definition 2.42) describes the double braiding: For all $g,h \in G$,

   $$b(g,h) = \psi(h,g)\psi(g,h) \ .$$

   Hint: Expand $\psi(gh,gh)$ into $\psi(g,gh)\psi(h,gh)$ and then further using the defining identities of an abelian 3-cocycle $(\omega,\psi) \in Z^3_{\mathrm{ab}}(G,k^\times)$. Check that the ω-factors one picks up along the way combine to a coboundary.

2. Show that θ_V in Example 5.5 (1) is a balancing isomorphism for $\mathcal{C}$.

3. Consider Vec_G^ω for G a possibly non-abelian group and ω a normalised 3-cocycle on G. Let $V = \bigoplus_{g \in G} V_g$ be a finite-dimensional G-graded vector space with dual V^* and evaluation map as in Example 5.5 (3).

Let $v_{g,i}$ be a basis of V_g and $v^*_{g,i}$ be the corresponding dual basis of $(V_g)^*$. Note that $|v_{g,i}| = g$ and $|v^*_{g,i}| = g^{-1}$. Show that

$$\begin{aligned} \mathrm{ev}_V : V^* \otimes V \to k \quad , &\qquad \varphi \otimes v \mapsto \varphi(v) \ , \\ \mathrm{coev}_V : k \to V \otimes V^* \quad , &\qquad 1 \mapsto \sum_{g,i} \omega(g, g^{-1}, g)\, v_{g,i} \otimes v^*_{g,i} \ , \end{aligned}$$

satisfy the zig-zag identities, and so V^* is indeed the left dual of V. *Hint:* You need to check both zig-zag identities, the computations are different. At some point you may need the identity $1 = (\delta\omega)(g^{-1}, g, g^{-1}, g)$.

4. Take the field $k = \mathbb{C}$ and consider $G = \mathbb{Z}_4$ with trivial abelian 3-cocycle $(\omega, \psi) = (1, 1)$. Give all balancing isomorphisms on $\big(\mathsf{Vec}_G^{(\omega,\psi)}\big)^{\mathrm{fd}}$ and state which of these define a ribbon category.

■ In a ribbon category $\mathcal{C}$ the left dual a^* of an object a is also a right dual via

$$\begin{aligned} \widetilde{\mathrm{ev}}_a &= \mathrm{ev}_a \circ \beta_{a,a^*} \circ (\theta_a \otimes 1_{a^*}) = \mathrm{ev}_a \circ \beta^{-1}_{a^*,a} \circ (\theta_a^{-1} \otimes 1_{a^*}) &&: a \otimes a^* \to \mathbb{1} \, , \\ \widetilde{\mathrm{coev}}_a &= (1_{a^*} \otimes \theta_a) \circ \beta_{a,a^*} \circ \mathrm{coev}_a = (1_{a^*} \otimes \theta_a^{-1}) \circ \beta^{-1}_{a^*,a} \circ \mathrm{coev}_a &&: \mathbb{1} \to a^* \otimes a \, . \end{aligned}$$

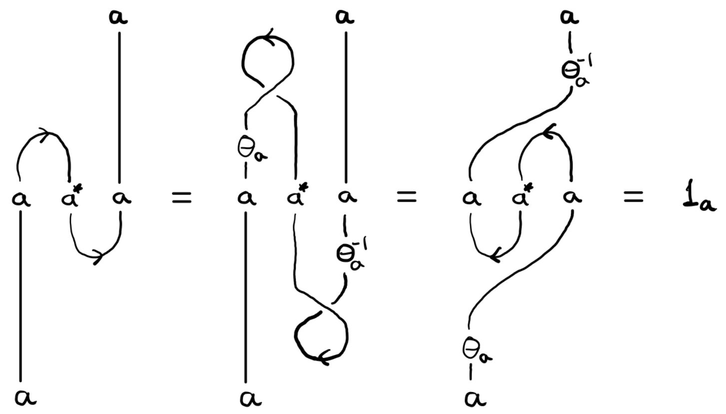
Note when we first constructed right duals from left duals in a braided setting (see above Proposition 2.33), we did not have the ribbon twists θ_a. For a symmetric monoidal category with its standard balancing by the identity natural transformation, this makes no difference. The reason to include θ_a here will become clear below.

To see that the two expressions given above are equal, one has to use the two defining identities of a ribbon twist. Namely, one starts from

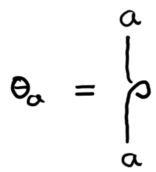
$$\mathrm{ev}_a = \theta_{\mathbb{1}} \circ \mathrm{ev}_a = \mathrm{ev}_a \circ \theta_{a^* \otimes a} = \mathrm{ev}_a \circ (\theta_{a^*} \otimes \theta_a) \circ \beta_{a,a^*} \circ \beta_{a^*,a}$$

and then brings one copy of the braiding and the ribbon twist to the other side and uses the property in Remark 5.2 (3). (Details?) To check the zig-zag identities, one can combine the first expression for $\widetilde{\mathrm{ev}}_a$ with the second ?

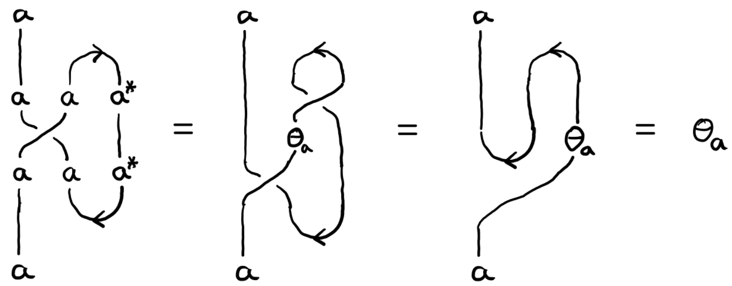
expression for $\widetilde{\mathrm{coev}}_a$, for example



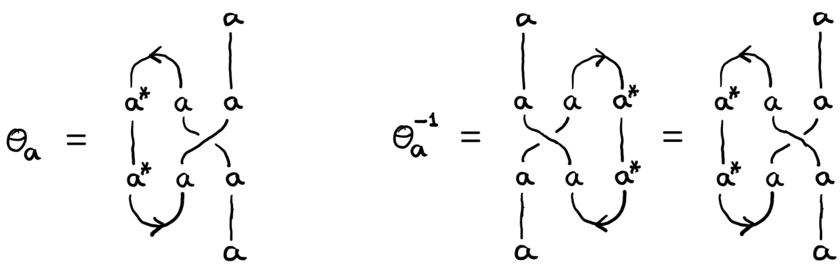
■ In a ribbon category $\mathcal{C}$ we use the following graphical notation for the ribbon twist:



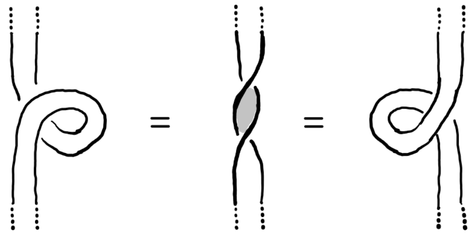
The notation is consistent, because



In fact we also have (Check this. For example, in the first identity, substitute the first expression for $\widetilde{\mathrm{coev}}_a$.) ?

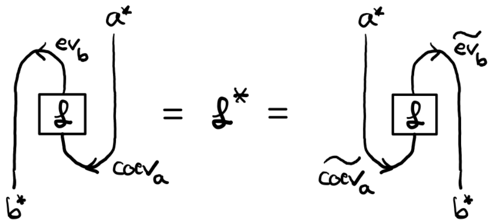


One can ask if there is a geometric object which naturally satisfies these identities. Unsurprisingly, given the name, the answer is yes: it is a ribbon in three-dimensional space:

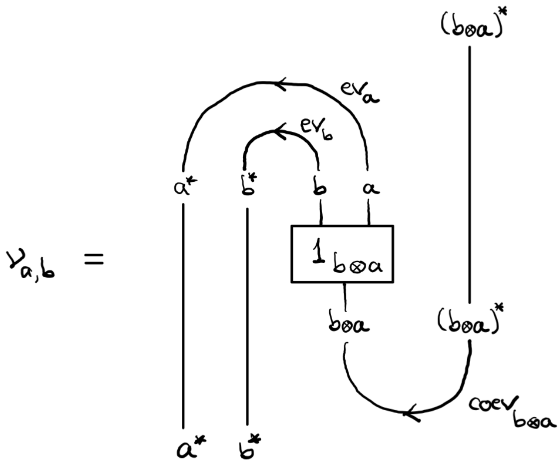


This is made precise in the next section.

■ Let $\mathcal{C}$ be ribbon category and let $f : a \to b$ be a morphism. As we said in the beginning of this section, there are in principle two notions of a dual morphism f^*, depending on whether one uses the (co)evaluation of the left or of the right duality. However, here they coincide: (Check this. *Hint:* The computation is similar to that of the zig-zag identity.) (?)



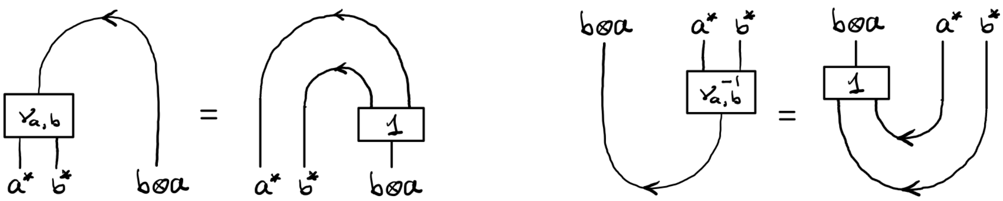
■ Define $\nu_{a,b} : a^* \otimes b^* \to (ba)^*$ to be the string diagram



The coupon with the identity morphism $1_{b\otimes a} : b \otimes a \to b \otimes a$ is an artefact of the graphical notation used to turn a single strand labelled "$b \otimes a$" into two strands, one labelled b and one labelled a. (Write out the (longish) expression for $\nu_{a,b}$ with all coherence isomorphisms.) The inverse $\nu_{a,b}^{-1} : (b\otimes a)^* \to a^* \otimes b^*$ (?)

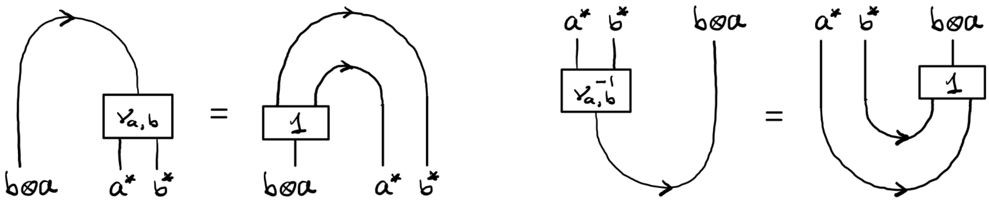
is given by rotating the string diagram above by 180° and adjusting the orientations. (Check this.) ?

From the zig-zag identity for the left duality it is clear that:



The next lemma shows that this is also true for the right duality morphisms.

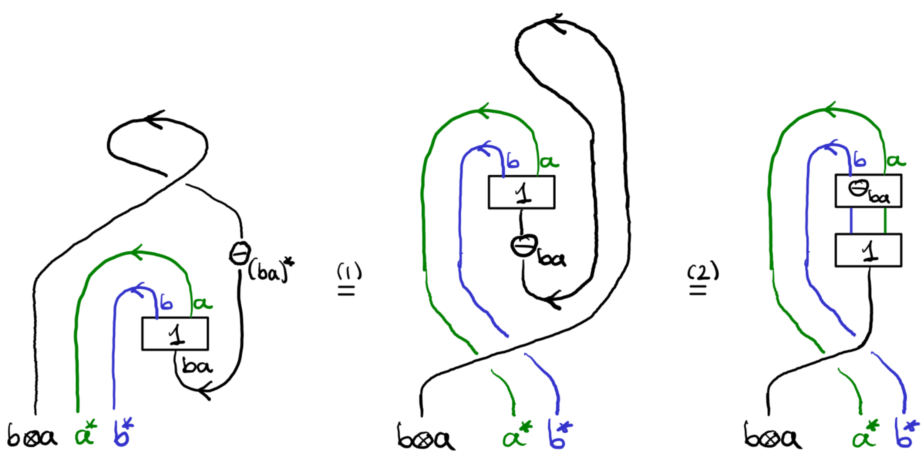
Lemma 5.7. We have

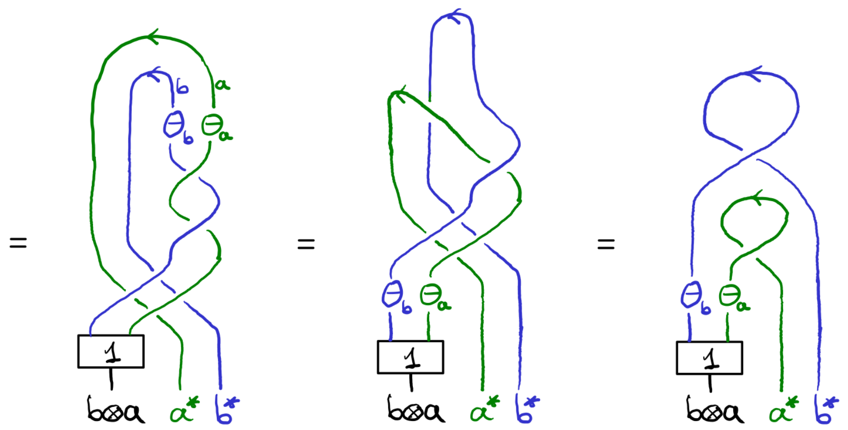


Proof. We only show the first identity, the second one then follows from uniqueness of the coevaluation map for a given evaluation map. We start from the left hand side and substitute yet another equivalent way to write the right evaluation map. Namely, for $x = b \otimes a$,

$$\widetilde{\mathrm{ev}}_x = \mathrm{ev}_x \circ \beta_{x,x^*} \circ (1_x \otimes \theta_{x^*}) \ .$$

This gives the first expression in:



$=$ 

In step 1 we use $(1_x \otimes \theta_{x^*}) \circ \mathrm{coev}_x = (\theta_x \otimes 1_{x^*}) \circ \mathrm{coev}_x$, for $x = b \otimes a$, as well as naturality of the braiding. Step 2 is a zig-zag identity, and we exchanged θ and the identity morphism. Then we substitute the defining relation in Definition 5.1 (1) and deform the diagram. $\square$

For info: The morphism ν is part of the monoidal structure on the contravariant functor $(-)^* : \mathcal{C} \to \mathcal{C}$ which one is always given if a monoidal category $\mathcal{C}$ has left duals (or right duals). Then the double dual functor $(-)^{**} : \mathcal{C} \to \mathcal{C}$ is a covariant monoidal functor. $\mathcal{C}$ is called *pivotal* if there is a natural monoidal isomorphism $\delta : \mathrm{Id}_{\mathcal{C}} \to (-)^{**}$, called the *pivotal structure*. An equivalent way to introduce pivotal structures is via the two properties above: left and right duals of morphisms agree and the identity in Lemma 5.7 holds. In particular, ribbon categories are pivotal. For more on pivotal structures see [EGNO, Sec. 4.7], and for the two equivalent characterisations see [CR1, Sec. 2.3].

■ For symmetric categories we defined the dimension $\dim(a)$ of an object a and showed in Proposition 2.33 that it is multiplicative. This generalises to ribbon categories.

We will consider dimensions as a special case of traces. Let $f : a \to a$ be a morphism in $\mathcal{C}$. Then the *trace of f* is

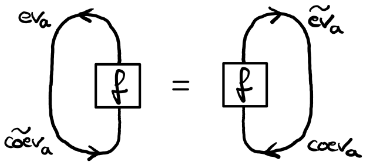
$$\mathrm{tr}_a(f) = \left[\mathbb{1} \xrightarrow{\widetilde{\mathrm{coev}}_a} a^* \otimes a \xrightarrow{1_{a^*} \otimes f} a^* \otimes a \xrightarrow{\mathrm{ev}_a} \mathbb{1} \right] .$$

(Check that for $\mathcal{C} = \mathsf{Vec}_k$, V a finite-dimensional k-vector spaces, and $f : V \to V$ a linear map, the above expression for $\mathrm{tr}_V(f)$ reduces to the standard definition of the trace of a linear endomorphism.) ? The *dimension of a* is the trace of the identity morphism,

$$\dim(a) = \mathrm{tr}_a(1_a) \in \mathrm{End}(\mathbb{1}) .$$

One can exchange the role of the left and right duality morphisms here without changing the trace:

Proposition 5.8. 1. For $f : a \to a$ we have



2. The trace is *cyclic*, that is, for $p : a \to b$ and $q : b \to a$,

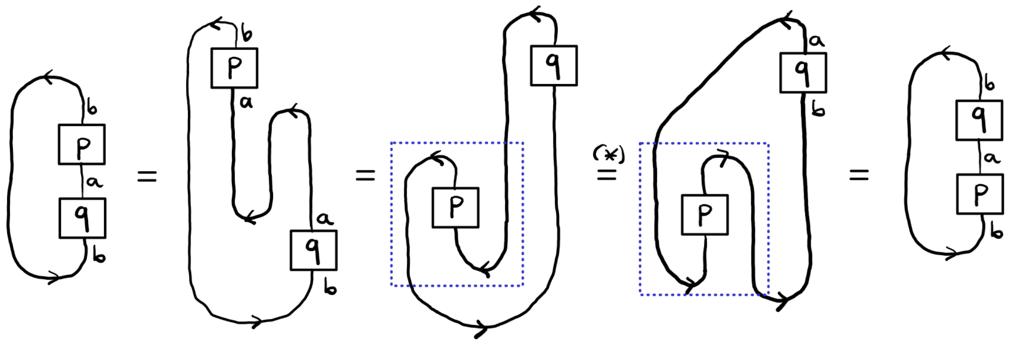
$$\mathrm{tr}_b(p \circ q) = \mathrm{tr}_a(q \circ p) \ .$$

3. For $f : a \to a$, $g : b \to b$ we have

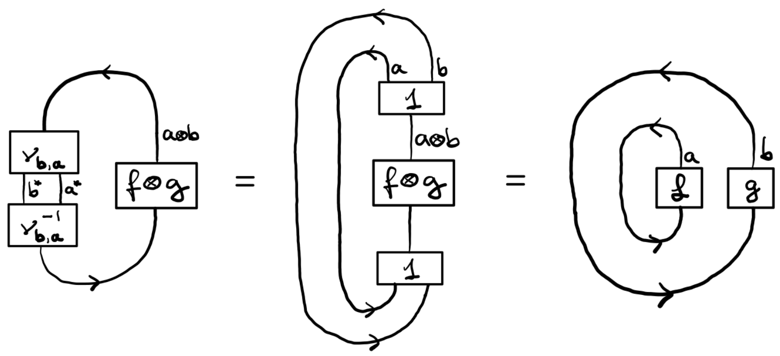
$$\mathrm{tr}_{a\otimes b}(f \otimes g) = \mathrm{tr}_a(f) \circ \mathrm{tr}_b(g) \ .$$

In particular, $\dim(a \otimes b) = \dim(a)\dim(b)$.

Proof. Part 1 is immediate from substituting the definition of the right duality maps and naturality of the ribbon twist. (Details?) For part 2, we compute ?



In step $(*)$ we use that the two ways of expressing the dual morphism f^* are equal. For part 3 use the identities for $\nu_{b,a}$ shown in and above Lemma 5.7:



□

For info: In general pivotal categories (which are not ribbon categories), the first identity in the above proposition need not be satisfied. In this case, one needs to distinguish left and right traces, and accordingly also left and right dimensions. Pivotal categories in which the first identity holds (i.e. where left traces are equal to right traces) are called *spherical*.

Exercise 5.9. Let $\mathcal{C}$ be a ribbon category. Rewrite $\theta_{a\otimes b}$ as a string diagram in terms of braiding and (co)evaluation. Write the (co)evaluation for $a\otimes b$ in terms of those for a and b. Show that the resulting diagram can be deformed to the diagram for $(\theta_a \otimes \theta_b)\circ \beta_{b,a}\circ\beta_{a,b}$. Use this as a way to remember the condition on the balancing isomorphism. E

Exercise 5.10. 1. Consider the ribbon category $\mathcal{C} = \big(\mathsf{Vec}_G^{(\omega,\psi,\chi)}\big)^{\mathrm{fd}}$ from Example 5.5 (3). E

Use the expressions for the left duality maps in Exercise 5.6 (3) to compute the right duality maps explicitly. With these, compute the dimension of $\mathbb{C}_g$, the one-dimensional vector space concentrated in degree $g\in G$.

Check explicitly that your expression for $\dim \mathbb{C}_g$ does not change if you replace $(\omega,\psi)\in Z^3_{\mathrm{ab}}(G,k^\times)$ by another normalised abelian 3-cocycle (ω',ψ') in the same cohomology class: $[\omega,\psi]=[\omega',\psi']$ in $H^3_{\mathrm{ab}}(G,k^\times)$.

2. Consider the additive group of complex numbers, $G=\mathbb{C}$. Set $\omega=1$ and $\psi(x,y)=\exp(i\pi xy)$. Show that there is no non-trivial group homomorphism $\chi:\mathbb{C}\to\{\pm1\}$. Set $\mathcal{C}=\big(\mathsf{Vec}_G^{(1,\psi,1)}\big)^{\mathrm{fd}}$.

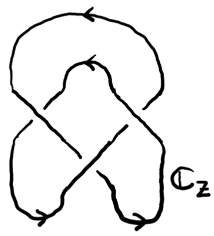
   Let $\mathbb{C}_z$ denotes the one-dimensional vector space in degree z. For $1\in\mathbb{C}_z$ denote by 1^* the corresponding dual basis element in $(\mathbb{C}_z)^*$.

   Compute the ribbon twist of $\mathbb{C}_z$. Compute the constants $\xi_{\varepsilon,\nu}$ in

$$\begin{aligned}
\beta_{\mathbb{C}_z,\mathbb{C}_z}(1\otimes 1) &= \xi_{++}\,1\otimes 1\ ,\\
\beta_{\mathbb{C}_z,(\mathbb{C}_z)^*}(1\otimes 1^*) &= \xi_{+-}\,1^*\otimes 1\ .\\
\beta_{(\mathbb{C}_z)^*,\mathbb{C}_z}(1^*\otimes 1) &= \xi_{-+}\,1\otimes 1^*\ ,\\
\beta_{(\mathbb{C}_z)^*,(\mathbb{C}_z)^*}(1^*\otimes 1^*) &= \xi_{--}\,1^*\otimes 1^*\ .
\end{aligned}$$

   What are the corresponding numbers for the inverse braiding?

3. Compute the value of the string diagram of the *left-hand trefoil knot*:



4. Look up the definition of the *linking number* of a two-component link, and of the *Tait number* of a regular knot diagram, for example in [Mur]. Relate these to the numbers one gets when evaluating string diagrams of the link or knot, with each object decorated by $\mathbb{C}_z$ as above.

5.2 The ribbon category of ribbon graphs

Let $\mathcal{C}$ be a ribbon category. We will first define a three-manifold with embedded ribbon graph as a three-manifold stratified in a very restrictive way. Then we consider specifically the three-manifold $\mathbb{R}^2 \times [0,1]$ with embedded ribbon graphs and will assemble these into a new ribbon category $\mathsf{Rib}_{\mathcal{C}}$. The main theorem in this section, which we state without proof, asserts that there is a unique ribbon functor $\mathsf{Rib}_{\mathcal{C}} \to \mathcal{C}$. This is the formal justification of the graphical calculus for ribbon categories.

■ The definition of a three-manifold with embedded ribbon graph will be reminiscent of a bordism for the n-dimensional trivial TFT with line defects from Section 4.4.

We take $D(\mathcal{C}) = D_0 \sqcup D_1 \sqcup D_2 \sqcup D_3$, with $D_0 = \mathrm{mor}(\mathcal{C}) \sqcup \{*_0\}$, the class of morphisms in $\mathcal{C}$, $D_1 = \mathrm{obj}(\mathcal{C}) \sqcup \{*_1\}$, the class of objects in $\mathcal{C}$, and $D_2 = \{*_2\}$, $D_3 = \{*_3\}$. The map $\pi : D \to \{0,1,2,3\}$ takes elements of D_k to k. The partial order is simply $D_0 < D_1 < D_2 < D_3$.

■ Recall from Section 4.4 the notation

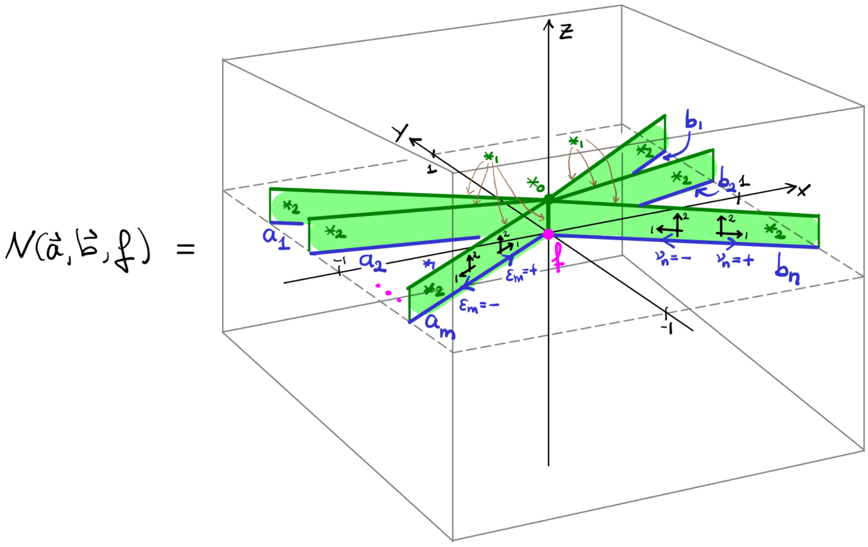
$$\vec{a} = \big(\,(a_1,\varepsilon_1),\ldots,(a_m,\varepsilon_m)\,\big)\ ,$$

with $a_i \in \mathcal{C}$ and $\varepsilon_i \in \{\pm\}$, and the assignment

$$T(\vec{a}) = a_1^{\varepsilon_1} \otimes (a_2^{\varepsilon_2} \otimes \cdots (a_{m-1}^{\varepsilon_{m-1}} \otimes a_m^{\varepsilon_m}) \cdots) \in \mathcal{C}\ .$$

where $a^+ = a$ and $a^- = a^*$. Let further $\vec{b} = ((b_1,\nu_1),\dots)$ and $f : T(\vec{a}) \to$

$T(\vec{b})$. Define $N(\vec{a}, \vec{b}, f)$ to be the cube $(-1,1)^3$, stratified as follows

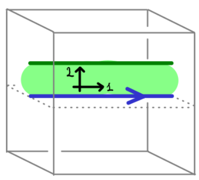


Here, the 0 stratum in the plane $\mathbb{R}^2 \times \{0\}$ is decorated by f and the 1-strata in that plane are decorated by the component objects of $\vec{a}$ and $\vec{b}$ as shown. The strata not in the plane $\mathbb{R}^2 \times \{0\}$ (i.e. those with positive z-coordinate) are decorated by $*_n$, $n = 0, 1, 2$, according to their dimension. The 1- and 2-strata are oriented, and their orientation depends on ε_i and ν_j as shown. The orientation of the strata labelled $*_1$ will not play a role and is not indicated.

Definition 5.11. A *3-manifold with embedded* $\mathcal{C}$*-ribbon graph* M is an oriented $D(\mathcal{C})$-stratified 3-manifold such that the $1,2$-strata in the interior are oriented, as are the $0,1$-strata on the boundary, and such that:

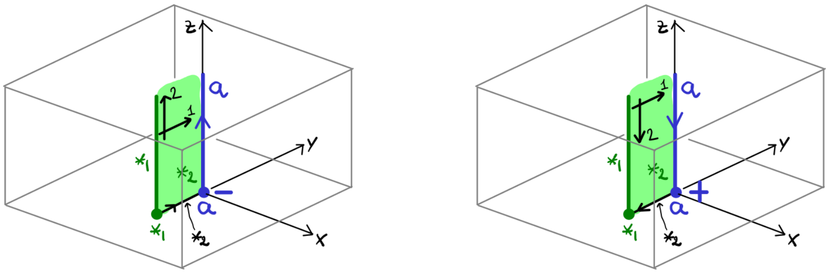
1. Each interior zero-stratum v not labelled by $*_0$ is equipped with an embedding $\varphi_v : N(\vec{a}, \vec{b}, f) \to M$ for some $\vec{a}, \vec{b}, f$, mapping the origin to v and respecting stratification and orientations.

2. Each interior point of M which is not a 0-stratum has a neighbourhood isomorphic to a cube $(-1,1)^3$ consisting of a single 3-stratum, or



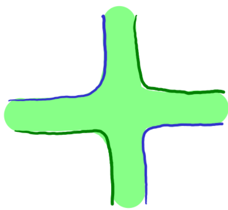
3. Each point on the boundary of M has a neighbourhood isomorphic to

$(-1,1)^2 \times [0,1)$ (all labelled $*_3$), or to $(-1,1)^2 \times [0,1)$ with stratifications



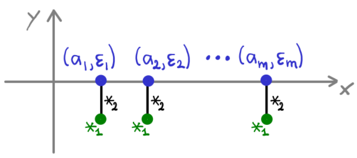
4. Denote the polyhedron formed by the $0,1,2$-strata in M by Γ, the *ribbon graph*. Let γ be the polyhedron formed by the 0 and 1 strata labelled by data of $\mathcal{C}$ (rather than by $*_n$). Then Γ is isomorphic to $\gamma \times [0,1]$.

Condition 4 implies that Γ retracts to γ, it excludes situations such as the stratification below, which would be allowed without condition 4.



(How does the midpoint of the 2-stratum in the above figure have a valid neighbourhood?) ?

■ For $\vec{a}$ of length m, let $\mathbb{R}^2_{\vec{a}}$ be the plane stratified as follows,



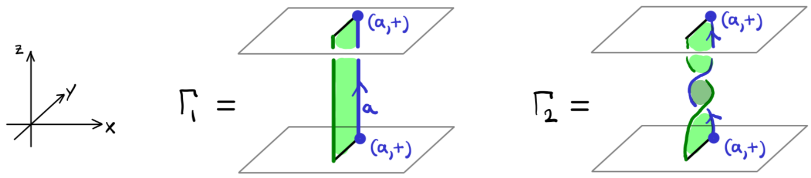
Given $\vec{a} = ((a_1,\varepsilon_1),(a_2,\varepsilon_2),\dots)$ write $-\vec{a} = ((a_1,-\varepsilon_1),(a_2,-\varepsilon_2),\dots)$. Note that this is not the same as the dual $(\vec{a})^*$, as in the latter we would also reverse the order of the list.

By a *ribbon graph* Γ *in* $\mathbb{R}^2 \times [0,1]$ *from* $\vec{a}$ *to* $\vec{b}$ we mean a ribbon graph in $\mathbb{R}^2 \times [0,1]$ with boundary $\mathbb{R}^2 \times \{0\}$ stratified as $\mathbb{R}^2_{-\vec{a}}$ and boundary $\mathbb{R}^2 \times \{1\}$ stratified as $\mathbb{R}^2_{\vec{b}}$. Write $\mathrm{Diag}(\vec{a},\vec{b})$ for the set of all such ribbon graphs.

We call $\Gamma_1, \Gamma_2 \in \mathrm{Diag}(\vec{a},\vec{b})$ *equivalent* if there is an isomorphism of stratified manifolds $\Gamma_1 \to \Gamma_2$ which is

- the identity on the boundary $\mathbb{R}^2 \times \{0,1\}$ of $\mathbb{R}^2 \times [0,1]$, and
- the identity outside $[-L, L]^2 \times [0,1]$ for L large enough.

The last condition rules out that we can rotate each slice $\mathbb{R}^2 \times \{t\}$ by some angle which would allow us for example to identify the two ribbon graphs $\Gamma_1, \Gamma_2 \in \mathrm{Diag}((a,+),(a,+))$ given by



which, as we will see below, would restrict us the the case where the ribbon twist is trivial and hence (by Remark 5.2 (2)) to the case that $\mathcal{C}$ is symmetric.

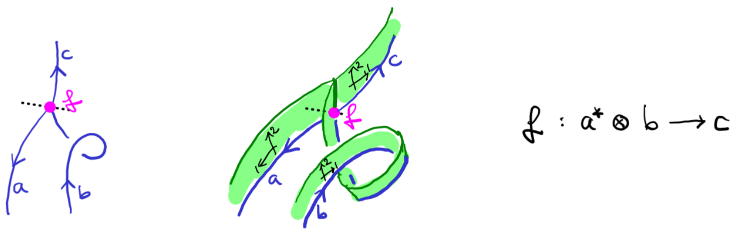
■ We can now finally give the definition of $\mathsf{Rib}_{\mathcal{C}}$.

Definition 5.12. Let $\mathcal{C}$ be a ribbon category. The category $\mathsf{Rib}_{\mathcal{C}}$ of $\mathcal{C}$-*coloured ribbon graphs* has

- *objects:* lists $\vec{a}$ with component objects $a_i \in \mathcal{C}$
- *morphisms $\vec{a} \to \vec{b}$:* equivalence classes of ribbon graphs $\Gamma \in \mathrm{Diag}(\vec{a}, \vec{b})$,
- *units $1_{\vec{a}}$:* equivalence class of $\mathbb{R}^2_{\vec{a}} \times [0,1]$,
- *composition:* For $\Gamma_1 \in \mathrm{Diag}(\vec{a}, \vec{b})$ and $\Gamma_2 \in \mathrm{Diag}(\vec{b}, \vec{c})$, we define $[\Gamma_2] \circ [\Gamma_1]$ by placing Γ_2 in $\mathbb{R}^2 \times [1,2]$ on top of Γ_1 and compressing back to $\mathbb{R}^2 \times [0,1]$ with by rescaling the z-coordinate with $\frac{1}{2}$.

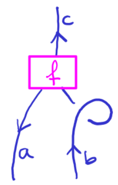
In the following, we will not distinguish a ribbon graph Γ and its equivalence class $[\Gamma]$ and write Γ for both.

Remark 5.13. To avoid drawing the full ribbon diagrams all the time, let us introduce the *paper plane convention.* There we arrange all ribbons to sit perpendicular on the paper plane (or just above or below in the case of a crossing), and so we cannot see the ribbon itself. For example



The paper plane as coordinates x, y and the z-axis is pointing towards us. The 2-orientation of the ribbon is chosen such that the orientation of the $\mathcal{C}$-labelled 1-stratum as first basis vector together with an inward pointing normal vector as second vector is positively oriented. Equivalently, if one rotates the ribbon with the left hand rule around the $\mathcal{C}$-labelled boundary edge into the paper plane, it is oriented as the paper plane.

We will often draw the 0-strata labelled by morphisms as coupons,



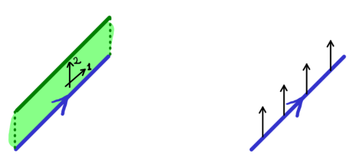
but this is just notation.

For info: 1. The more standard description of a ribbon graph, as for example used in [Tu, Sec. I.2.1], is as embedded ribbons and coupons. A coupon would be a square $[0,1]^2$ embedded in $\mathbb{R}^2 \times [0,1]$, and a ribbon an embedded rectangle $[0,1] \times [0, \frac{1}{10}]$ or annulus $A = S^1 \times [0, \frac{1}{10}]$ (of course the $\frac{1}{10}$ is immaterial, it just visualises that the ends of a ribbon are $\{0\} \times [0, \frac{1}{10}]$ and $\{1\} \times [0, \frac{1}{10}]$. The ends have to be attached to coupons or the boundary of $\mathbb{R}^2 \times [0,1]$, see [Tu, Sec. I.2.1] for the full details.

To get from our description to that in [Tu] one needs to rotate our ribbons by 90° as in the paper plane convention above, and replace the 0-strata labelled by morphisms with coupons. This procedure is well-defined up to equivalence, and is the reason why we can appeal to the results in [Tu] later on.

2. The reason why we picked a different convention is that from the point of view of defect TFTs, if one assigns objects in a ribbon category to 1-strata, it is natural to assign morphisms to 0-strata, and not to coupons. This becomes even more apparent in the smooth setting briefly discussed in the next point.

3. If we were using smooth manifolds, we could consider embedded framed graphs, which correspond to ribbons as follows,

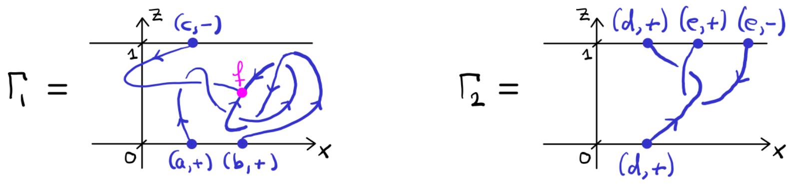


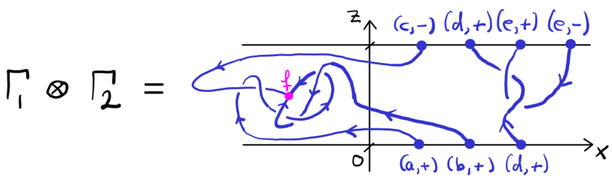
The framed lines can then meet at framed points, with suitable compatibility conditions for the framing. In this sense, we are developing a "3d TFT with framed line and point defects".

■ The category $\mathsf{Rib}_\mathcal{C}$ is itself a ribbon category:

1. *tensor product:* For ribbon graphs $\Gamma_1 : \vec{a} \to \vec{a}'$ and $\Gamma_2 : \vec{b} \to \vec{b}'$, we "place the two ribbon graphs next to each other" to get a ribbon graph $\Gamma_1 \otimes \Gamma_2 : \vec{a} * \vec{b} \to \vec{a}' * \vec{b}'$. If Γ_i is contained in $(-L_i, L_i) \times \mathbb{R} \times [0,1]$, then place Γ_1 at x-coordinates $(-2L_1, 0)$ and Γ_2 at $(0, 2L_2)$, and then move the boundary strata back to standard position.

   This is tedious to write down in full detail, but because we are using equivalence classes of ribbon graphs, the precise procedure is also not important. Here is an example:

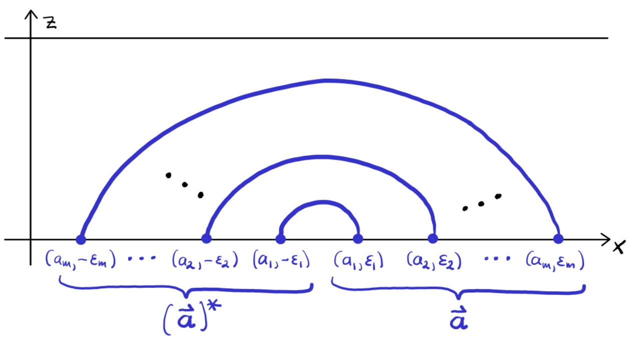




   The tensor unit is the empty list $\mathbb{1} = ()$.

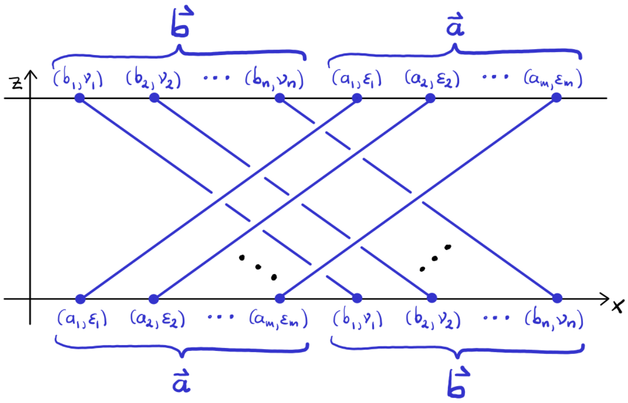
   Note that with this tensor product and tensor unit, $\mathsf{Rib}_\mathcal{C}$ is strict monoidal.

2. *left duality:* The left dual of $\vec{a} = ((a_1, \varepsilon_1), \ldots, (a_m, \varepsilon_m))$ is $(\vec{a})^* = ((a_m, -\varepsilon_m), \ldots, (a_1, -\varepsilon_1))$, with evaluation morphisms

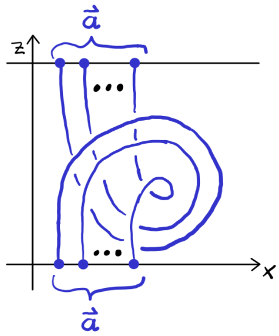


   where the orientation of the ribbons is determined by the signs ε_i. The coevaluation morphism $() \to \vec{a} \otimes (\vec{a})^*$ is similar.

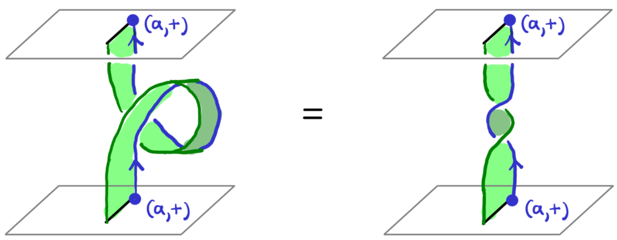
3. *braiding:*



4. *balancing:*



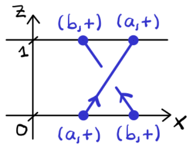
The above ribbon graphs are drawn in the paper-plane convention. For example, if $\vec{a} = ((a, +))$, in perspective the balancing isomorphism looks like

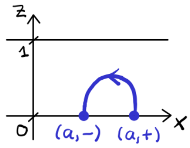


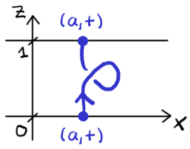
Exercise 5.14. Check the defining relations of a ribbon category by drawing convincing diagrams. *Hint:* Invent a notation which allows you to collect the ribbons attached to $\vec{a}$ into one graphical representation, e.g. into a square times an interval, with the ribbons intersecting the square in the intervals $\{t_1\} \times [0, 1] \subset [0, 1] \times [0, 1]$ (E)

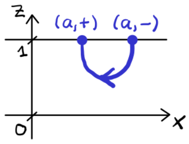
■ An *elementary ribbon graph* is a ribbon graph Γ_f isomorphic to the $N(\vec{a}, \vec{b}, f)$.

Define furthermore the ribbon graphs

$X_{(a,+),(b,+)} =$ 

$\cap_{(a,+)} =$ 

$t_{(a,+)} =$ 

$\cup_{(a,+)} =$ 

We can now state the main theorem for $\mathsf{Rib}_\mathcal{C}$, see [Tu, Thm. I.2.5]. We will not prove the theorem, but instead refer to the proof via generators and relations in [Tu, Ch. I].

Theorem 5.15. Let $\mathcal{C}$ be a ribbon category. There is a unique monoidal functor

$$F_\mathcal{C} : \mathsf{Rib}_\mathcal{C} \to \mathcal{C} \ ,$$

called the *Reshetikhin-Turaev functor*, such that

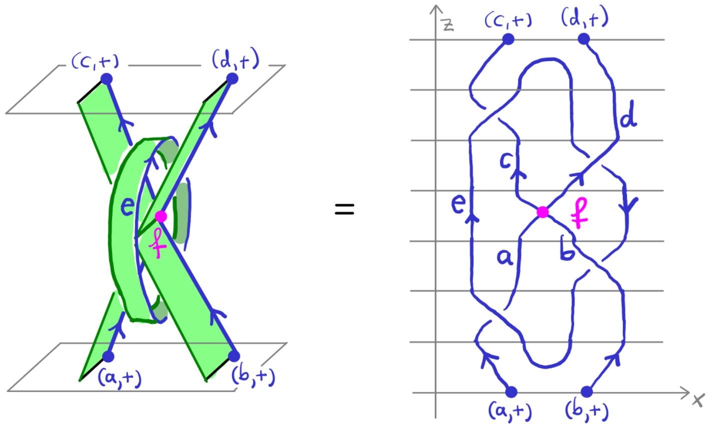
1. $F_\mathcal{C}(\vec{a}) = T(\vec{a}) \in \mathcal{C}$,
2. $F_\mathcal{C}(\Gamma_f) = f$ for an elementary ribbon graph Γ_f,
3. We have

$$F_\mathcal{C}(X_{(a,+),(b,+)}) = \beta_{a,b} \qquad F_\mathcal{C}(\cap_{(a,+)}) = \mathrm{ev}_a$$
$$F_\mathcal{C}(t_{(a,+)}) = \theta_a \qquad F_\mathcal{C}(\cup_{(a,+)}) = \mathrm{coev}_a$$

The unique monoidal functor $F_\mathcal{C}$ is a ribbon functor. (9.6.26)

■ To evaluate $F_\mathcal{C}$ isotope the ribbon graph such that its projection onto the x-z-plane has the following property: There is a finite number of points $0 = z_0 < z_1 < z_2 < \cdots < z_{n-1} < z_n = 1$ such that in each interval $[z_j, z_{j+1}]$ there is one of the ribbon graphs listed in conditions 2 or 3 above, together with possibly straight vertical ribbons to its left and right, in paper-plane framing relative to the x-z-plane. Using functoriality of $F_\mathcal{C}$, one can then write $F_\mathcal{C}(\Gamma)$ as a composition of identity morphisms tensored with one of the morphisms of $\mathcal{C}$ listed in condition 2 or 3.

For example,



Exercise 5.16. Let $\mathcal{C}$ be a ribbon category. E

1. Let $\Gamma \in \mathrm{Diag}(\vec{a}, \vec{b})$ be a ribbon graph, and let A be a component of Γ isomorphic to an embedded annulus labelled by $x \in \mathcal{C}$. Let Γ' be obtained from Γ by changing the orientation of the 1-stratum labelled x in A and replacing the label by x^*. Show that $F_{\mathcal{C}}(\Gamma) = F_{\mathcal{C}}(\Gamma')$. *Hint:* Consider the coupon (0-stratum) with one incoming and one outgoing 1-stratum, both oriented away from the 0-stratum, the incoming one labelled x and the outgoing one labelled x^*. Then the 0-stratum is labelled by a morphism $x^* \to x^*$, and we choose the identity 1_{x^*}. Do the same for a 0-stratum with both 1-strata pointing towards the coupon, with the incoming one labelled x^* and the outgoing x. Show that the pair cancels in any order.

2. For $a, b \in \mathcal{C}$ define the S-matrix
$$S_{a,b} = \mathrm{tr}_{a\otimes b}(\beta_{b,a} \circ \beta_{a,b}) \ .$$
Look up a picture of a Hopf-link. Show that $S_{a,b}$ can be written as $F_{\mathcal{C}}$ applied to a suitably coloured and oriented framed Hopf link. Use this representation to show by isotopies of the framed Hopf link that
$$S_{a,b} = S_{b,a} = S_{a^*,b^*} \quad \text{and} \quad S_{a,b^*} = \mathrm{tr}_{a\otimes b}(\beta_{a,b}^{-1} \circ \beta_{b,a}^{-1}) \ .$$

■ By construction, $F_{\mathcal{C}}$ is invariant under isotopies (isomorphisms which are the identity outside a compact region). A *framed $\mathcal{C}$-coloured link* is a ribbon graph $() \to ()$ without coupons. Thus, via $F_{\mathcal{C}}$, a ribbon category $\mathcal{C}$ produces invariants of framed links. To get invariants of unframed links, one needs to colour the link with an object $a \in \mathcal{C}$ with trivial ribbon twist, $\theta_a = 1_a$.

For the invariant to be interesting, the ribbon twist cannot be trivial for all objects, for else the category $\mathcal{C}$ is symmetric, and in that case the invariant of a link with n components coloured by $a_1, \dots, a_n$ is simply $\prod_{i=1}^n \dim(a_i)$. We will not show this in detail, but one way to see this is to note that for $\mathcal{C}$ symmetric, we can factor $F_\mathcal{C}$ through the functor L from Section 4.4 and a closed one-dimensional bordism is determined by its number of components.

For info: The relation to link invariants raises a number of questions:

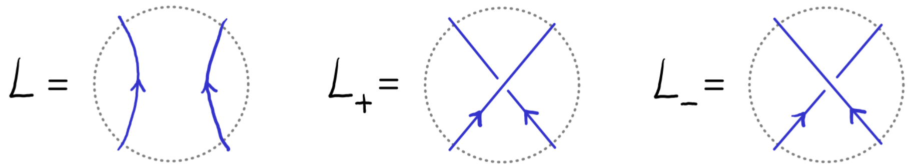
- Is there a ribbon category $\mathcal{C}$ and a choice of object $a \in \mathcal{C}$ such that $F_\mathcal{C}$ is a complete invariant (i.e. can distinguish all framed links with all components coloured a)?
- Given two non-isotopic framed knots K, K' is there a ribbon category $\mathcal{C}$ and a choice of object $a \in \mathcal{C}$ such that $F_\mathcal{C}(K_a) \neq F_\mathcal{C}(K'_a)$?
- How do we find explicit examples of ribbon categories that give interesting knot invariants?

The answers to 1 (and hence 2) is trivially "yes" because one can just take the geometric category $\mathsf{Rib}_\mathcal{C}$ for any $\mathcal{C}$. So we need to modify questions 1 and 2 to make the input ribbon category algebraic in the sense that the underlying category is a semisimple category or at least locally finite abelian (see [EGNO, Sec. 1.8]). In that case the answer to 1, or even to the weaker statement 2, is not known. We will instead look at question 3 and try to give interesting examples of algebraic ribbon categories.

Exercise 5.17. Let $\mathcal{C}$ be a k-linear ribbon category such that the endomorphism of $\mathbb{1}$ are a one-dimensional vector space, $\mathrm{End}(\mathbb{1}) = k 1_\mathbb{1}$. Let us assume for simplicity that $\mathcal{C}$ is strict monoidal. Suppose $J \in \mathcal{C}$ is tensor-invertible (see above Lemma 4.20). E

1. Show that also $\mathrm{End}(J) = k\,1_J$. *Hint:* Pick L such that $J \otimes L \cong \mathbb{1}$. Show that $\mathrm{End}(J) \to \mathrm{End}(J \otimes L)$, $f \mapsto f \otimes 1_L$ is injective.
2. Show that in part 1, $L \cong J^*$. *Hint:* As L is tensor-invertible as well, also $\mathrm{End}(L) = k\,1_L$. Pick isomorphism $\phi : L \otimes J \to \mathbb{1}$ and $\psi : \mathbb{1} \to J \otimes L$. Show that both "zig-zag identities" hold up to non-zero constants. Show that the map $L \to J^*$ obtained from ϕ is an isomorphism.
3. Since J is tensor-invertible, so is $J \otimes J$. Hence also $\mathrm{End}(J \otimes J)$ is one-dimensional. Define constants $t, b, d \in k^\times$ by
$$\theta_J = t\,1_J \ , \quad \dim(J) = d\,1_\mathbb{1} \ , \quad \beta_{J,J} = b\,1_{J \otimes J} \ .$$
This implies $\beta_{J,J}^{-1} = b^{-1}$. Show that
$$d^2 = 1 \quad , \quad t = d\,b \ .$$

4. Let L be a framed link with all components labelled by J. Suppose we get the framed links $L_\pm$ from L by changing L inside a 3-ball as below, but keeping it the same outside of the three-ball:



Show that $F_{\mathcal{C}}(L_\pm) = b^{\pm 1} F_{\mathcal{C}}(L)$. Use this to argue that invariants for two-component links obtained from graded vector spaces in this way can be expressed in terms of linking number and Tait number of the components in the corresponding link diagram.

5.3 Fusion categories

Let k be a field and A a k-algebra. As before (cf. Example 2.8 (3)), we write A-mod for the category of left A-modules (over k, i.e. each A-module M is in particular a k-vector space). Note that A-mod is k-linear.

An A-module M is *simple* if it is non-zero and if it has no submodules other that $\{0\}$ and M. (At this point it is good to recall (or look up) how to prove the following statements: Let M be simple and N an arbitrary A-module. An A-module morphism $f : M \to N$ is either zero or injective. An A-module morphism $g : N \to M$ is either zero or surjective. If also N is simple, then an A-module morphism $h : M \to N$ is either zero or an isomorphism. *Hint:* The kernel and the image of A-module morphisms are submodules. The key word is "Schur's Lemma", even though that specific statement is not it.) A k-algebra A is *semisimple* if each A-module is a direct sum of simple A-modules.

Example 5.18. 1. $A = k^n = k \oplus k \oplus \cdots \oplus k$ with pointwise multiplication and addition is a semisimple k-algebra. If we set $I = \{1, 2, \ldots, n\}$, then A-$\mathsf{mod} = \mathsf{Vec}_I$, the category of vector spaces graded by the set I. The simple objects are k_i, $i = 1, \ldots, n$, that is, the 1-dimensional k-vector space k, concentrated in degree i. (What is the action of A on a given I-graded vector space?)

2. $A = k[X]/\langle X^2 \rangle$ is not semisimple. The unique-up-to-isomorphism simple A-module is k with X acting as 0. (Why? *Hint:* Let $\rho(X) : M \to M$ be the action of X on an A-module M. The $\ker \rho(X)$ is a submodule. Why is $\ker \rho(X) = \{0\}$ not possible?) However, the regular A-module A is not a direct sum of copies of k (as X acts non-trivially on A).

Definition 5.19. Let D be a k-linear category. Then D is called:

1. *finite abelian* if it is equivalent, as a k-linear category, to the category $A\text{-}\mathsf{mod}^{\mathrm{fd}}$ of finite-dimensional A-modules for a finite-dimensional k-algebra A.

2. *finitely semisimple* if the algebra A in part 1 can be chosen to be semisimple.

Remark 5.20. 1. A finite abelian category has only finitely many isomorphism classes of simple objects. This follows from the corresponding statement for finite-dimensional A-modules: The regular A-module is finite-dimensional, hence has a finite composition series. But A also has a surjective module homomorphism to every simple module, so every simple module is a composition factor.

2. The endomorphism algebra of a simple object in a finite abelian category is a finite-dimensional division algebra over k. Indeed, every endomorphism of a simple module is either zero or an isomorphism. Over $\mathbb{R}$, there are three finite-dimensional division algebras: $\mathbb{R}$, $\mathbb{C}$, and the quaternions $\mathbb{H}$. If k is algebraically closed, then the endomorphisms of a simple object are always given by k.

For info: The usual approach to finite semisimplicity would be to first define additive categories, then say when an additive category is abelian (has kernels and cokernels, and every mono is a kernel and every epi is a cokernel), see [ML, Ch. VIII]. An abelian category is semisimple if every object is a direct sum (i.e. a categorical coproduct) of simple objects.

A k-linear abelian category is finite if it has finite-dimensional Hom-spaces, every object has finite composition length, there are enough projectives, and there are finitely many isomorphism classes of simples, see [EGNO, Sec. 1.8] for this formulation, as well as the equivalent one given above.

■ Let D be a k-linear category. For objects $a_1, \dots, a_n$ in D, their *direct sum* is an object d, together with maps $e_i : a_i \to d$, $r_i : d \to a_i$, such that

$$r_i \circ e_j = \delta_{i,j} 1_i \quad , \quad \sum_{i=1}^{n} e_i \circ r_i = 1_d \ .$$

Instead of d we write $\bigoplus_i a_i$ for the direct sum and suppress the maps e_i, r_i in the notation (but they are part of the data of a direct sum). A direct sum is unique up to unique isomorphism, and we will hence say "the direct sum" (Make the uniqueness statement precise and prove it. *Hint:* Start with "Let d', e_i', r_i' be another direct sum of the a_i. Then ...") ?

Recall from Definition 2.1 the notation $D(x, y) = \mathrm{Hom}_D(x, y)$.

Lemma 5.21. Let d, e_i, r_i be a direct sum of $a_1, \dots, a_n \in D$ as above. Then

$$\bigoplus_{i=1}^{n} D(a_i, x) \to D(d, x) \ , \quad (f_i)_i \mapsto \sum_{i=1}^{n} f_i \circ r_i$$

is an isomorphism of $\Bbbk$-vector spaces. There is a similar decomposition for $D(x, d)$. (What does the explicit formula look like?) ?

Proof. The inverse map $D(d, x) \to \bigoplus_{i=1}^{n} D(a_i, x)$ is $g \mapsto (g \circ e_i)_i$. One checks that this is indeed the inverse, for example

$$g \mapsto (g \circ e_i)_i \mapsto \sum_i g \circ e_i \circ r_i = g \circ 1_d = g \ .$$

□

■ The zero object $0 \in D$ is the direct sum with zero summands. In A-mod it is the A-module $\{0\}$. (For info: In general, by definition a zero object (or null object) is an object that is both initial and terminal, cf. [ML, Sec. I.5].)

Definition 5.22. A *fusion category* over an algebraically closed field k is a k-linear rigid monoidal category such that

- it is finitely semisimple as a k-linear category,
- the tensor product functor is k-linear in each argument,
- the tensor unit $\mathbb{1}$ is a simple object.

The reason to demand that k is algebraically closed is Remark 5.20 (2): If a is a simple object in a fusion category, then $\mathrm{End}(a) = k\, 1_a$.

For info: More generally, one defines [EGNO, Sec. 4.1]

1. A *finite tensor category* has the same definition as above, with "finitely semisimple" replaced by "finite abelian".
2. If the tensor unit is not simple, one instead speaks of a multitensor or multifusion category.

We will only consider fusion categories and not finite tensor categories. Yes, the course is called "tensor categories and ...", but the title refers more broadly to various forms of monoidal categories, and "finite tensor category" in [EGNO] has a very specific technical meaning.

Example 5.23. Let k be an algebraically closed field and G a finite group.

1. For $\omega \in Z^3(G, k^\times)$ a normalised 3-cocycle, $\left(\mathsf{Vec}_G^\omega\right)^{\mathrm{fd}}$ is a fusion category.

2. If $|G| \neq 0$ in k, then $\mathrm{Rep}(G)^{\mathrm{fd}}$, the category of finite-dimensional representations of G, with tensor product as in Example 2.17 (4), is a fusion category. (What is the left and right dual? How does G act on it?). ? The algebra is the group algebra $k[G]$, which is separable by Example 4.37 (3) and by the info point after Exercise 4.39, separable implies semisimple. (For info: The converse is false, in general. But it is true if, e.g. the field over which one is working is algebraically closed.)

Note that in both examples, the forgetful functor $U : C \to \mathsf{Vec}_k$ satisfies $U(X \otimes Y) \cong U(X) \otimes_k U(Y)$, naturally in X, Y. The isomorphisms may not satisfy the compatibility with the associator in Definition 2.23 (and cannot satisfy it in the first example if $[\omega]$ is non-trivial in $H^3(G, k^\times)$). Such a functor is called a *quasi-tensor functor*. One can ask if there are fusion categories for which there is no quasi-tensor functor to Vec_k at all. The answer is yes, and we will construct two examples, but it takes some preparation.

■ For the rest of this section, we fix

- an algebraically closed field $\Bbbk$,
- a fusion category $\mathcal{C}$ over $\Bbbk$,
- a choice $\mathcal{I}$ of representatives of the isomorphism classes of simple objects of D, such that $\mathbb{1} \in \mathcal{I}$.

The last choice means that if $x \in D$ is simple, then there exists a unique $a \in \mathcal{I}$ such that $a \cong x$. Note that by Remark 5.20 (1), the set $\mathcal{I}$ is finite.

We would like to describe the tensor product, associator and unitors of a fusion category "in a basis", that is, in terms of a finite collection of numbers (in $\Bbbk$). We will first extract these numbers ("fusion rules" and "F-matrices") and then see that they determine a fusion category up to monoidal equivalence. This will take some preparation.

■ As $\mathcal{C}$ is finitely semisimple, any $X \in \mathcal{C}$ can be written as a finite direct sum

$$X = \bigoplus_{a \in \mathcal{I}} a^{\oplus n_a} \ ,$$

where $a^{\oplus n_a} = a \oplus \cdots \oplus a$, the n_a-fold direct sum of a with itself. Then n_a is uniquely determined by X via

$$\dim_\Bbbk \mathcal{C}(a, X) \ = \ n_a \ = \ \dim_\Bbbk \mathcal{C}(X, a) \ .$$

To see this, first note that, for $a, b \in \mathcal{I}$,

$$\mathcal{C}(a,b) \;=\; \begin{cases} \Bbbk\, 1_a & ; a = b \;, \\ \{0\} & ; a \neq b \;. \end{cases}$$

Indeed, if $a \neq b$, then by construction they are not isomorphic, and as both are simple, the only morphism between them is the zero morphism.

Next, combine this with Lemma 5.21. For example the second identity follows from

$$\mathcal{C}(X,a) \;\cong\; \bigoplus_{b \in \mathcal{I}} \mathcal{C}(b,a)^{\oplus n_b} \;\cong\; \mathcal{C}(a,a)^{\oplus n_a} \;\cong\; \Bbbk^{\oplus n_a} \;.$$

■ Define the pairing

$$\kappa : \mathcal{C}(X,a) \times \mathcal{C}(a,X) \to \Bbbk \;,$$

for $X \in \mathcal{C}$ and $a \in \mathcal{I}$, to be the unique number $\kappa(f,g) \in \Bbbk$ such that

$$f \circ g = \kappa(f,g) 1_a \;\in \mathrm{End}(a) \;.$$

Here we are again using that $\Bbbk$ is algebraically closed, so that we necessarily have $\mathrm{End}(a) = \Bbbk\, 1_a$.

Exercise 5.24. Let $X \in \mathcal{C}$ with decomposition $X = \bigoplus_{a \in \mathcal{I}} a^{\oplus n_a}$, and let (E)

$$e_{a,\alpha} : a \to X \;,\quad r_{a,\alpha} : X \to a \quad,\quad \text{where} \quad \alpha = 1, \dots, n_a$$

be the embedding and restriction morphisms for the α's summand in $a^{\oplus n_a}$:

$$r_{a,\alpha} \circ e_{a,\beta} = \delta_{\alpha,\beta}\, 1_a \quad , \quad \sum_{a \in \mathcal{I}} \sum_{\alpha=1}^{n_a} e_{a,\alpha} \circ r_{a,\alpha} = 1_X \;.$$

(Aside: For finite-dimensional vector spaces with have $\mathcal{I} = \{\Bbbk\}$ and choosing e, r for a finite-dimensional vector space X as above is exactly the same as choosing a basis of X and the corresponding dual basis of X^*.)

1. Show that $\{e_{a,\alpha}\}$ is a basis of the $\Bbbk$-vector space $\mathcal{C}(a,X)$, and that $\{r_{a,\alpha}\}$ is a basis of $\mathcal{C}(X,a)$. Show that the pairing κ is non-degenerate.

2. Let $Y \in C$ and $f : X \to Y$. Let $\tilde{e}_{a,\alpha} : a \to Y$, $\tilde{r}_{b,\alpha} : Y \to b$, $\beta = 1, \dots, \tilde{n}_b$ be the corresponding maps which exhibit Y as a direct sum $Y = \bigoplus_{b \in \mathcal{I}} b^{\oplus \tilde{n}_b}$. Show that there are constants $f_{a;\beta,\alpha} \in \Bbbk$ such that

$$f = \sum_{a \in \mathcal{I}} \sum_{\alpha=1}^{n_a} \sum_{\beta=1}^{\tilde{n}_a} f_{a;\beta,\alpha}\, \tilde{e}_{a,\beta} \circ r_{a,\alpha}$$

Give an explicit expression for $f_{a;\beta,\alpha}$ in terms of κ.

3. Let $X, Y \in \mathcal{C}$, $f : X \to Y$. Show that, if $f \circ e = 0$ for all $e : a \to X$, $a \in \mathcal{I}$, then $f = 0$. Show also that if $r \circ f = 0$ for all $r : Y \to a$, $a \in \mathcal{I}$, the $f = 0$.

4. Show that for all $X, Y \in \mathcal{C}$ we have $\mathcal{C}(X, Y) \cong \mathcal{C}(Y, X)$.

5. Let now $\{v_{a,\alpha}\}$ be an arbitrary basis of $\mathcal{C}(X, a)$. Let $\bar{v}_{a,\alpha}$ be the dual basis of $\mathcal{C}(a, X)$ with respect to κ in the sense that $\kappa(v_{a,\alpha}, \bar{v}_{a,\beta}) = \delta_{\alpha,\beta}$. Show that also

$$e'_{a,\alpha} := \bar{v}_{a,\alpha} : a \to X \ , \quad r'_{a,\alpha} := v_{a,\alpha} : X \to a \quad , \quad \text{where} \quad \alpha = 1, \dots, n_a$$

exhibit X as the direct sum $\bigoplus_{a \in \mathcal{I}} a^{\oplus n_a}$.

6. Let now $\mathcal{C}$ be a fusion category which is in addition ribbon. Show that for $X, Y \in \mathcal{C}$, the *trace pairing*

$$t : \mathcal{C}(X, Y) \times \mathcal{C}(Y, X) \to \Bbbk \quad , \quad (f, g) \mapsto \mathrm{tr}_Y(f \circ g) \ ,$$

is non-degenerate. You may assume that $\dim(a) \neq 0$ for all $a \in \mathcal{I}$. (For info: To show this, one uses that $\mathbb{1}$ occurs with multiplicity one in $a^* \otimes a$, and since ev_a and $\widetilde{\mathrm{coev}}_a$ are nonzero and map between $\mathbb{1}$ and the summand $\mathbb{1}$ in $a^* \otimes a$, their composition is non-zero, too. See [EGNO, Prop. 4.8.4] for this argument.).

■ In a k-linear (or even just additive) monoidal category, the tensor product distributes over direct sums: Let M be a monoidal category, and let $x_1, x_2, y \in M$. Then $(x_1 \oplus x_1) \otimes y \cong x_1 \otimes y \oplus x_2 \otimes y$, and analogously for the second tensor factor. Indeed, let e_i, r_i, $i = 1, 2$ realise the direct sum $x_1 \oplus x_2$. Then $e_i \otimes 1_y$, $r_i \otimes 1_y$ exhibit $(x_1 \oplus x_2) \otimes y$ as direct sum of $x_1 \otimes y$ and $x_2 \otimes y$.

Lemma 5.25. Let $X, Y \in \mathcal{C}$. If $X \otimes Y = 0$, then $X = 0$ or $Y = 0$.

Proof. We will show that if $X \neq 0$ and $Y \neq 0$, then also $X \otimes Y \neq 0$. We first check the claim for $X = a$ and $Y = b$, with $a, b \in \mathcal{I}$.

Suppose $a \otimes b = 0$. Then also $(a^* \otimes a) \otimes b \cong a^* \otimes (a \otimes b) = 0$. But the morphism $\mathrm{ev}_a : a^* \otimes a \to \mathbb{1}$ is nonzero (as $a \neq 0$, the ziq-zag identities require ev_a and coev_a to be nonzero) and $\mathbb{1}$ is simple, and so $\mathbb{1}$ is a direct summand of $a^* \otimes a$: $a^* \otimes a \cong \mathbb{1} \oplus Q$, for some $Q \in \mathcal{C}$. But then also

$$(a^* \otimes a) \otimes b \cong \mathbb{1} \otimes b \oplus Q \otimes b \cong b \oplus Q' \neq 0 \ .$$

To show the statement of the lemma, suppose that X, Y are both nonzero. Writing $X = \bigoplus_{a \in \mathcal{I}} a^{\oplus n_a}$ and $Y = \bigoplus_{b \in \mathcal{I}} b^{\oplus \tilde{n}_b}$, then there are a_0, b_0 with $n_{a_0}, \tilde{n}_{b_0} > 0$. But then

$$X \otimes Y \cong \bigoplus_{a,b \in \mathcal{I}} (a \otimes b)^{n_a \, \tilde{n}_b} = (a_0 \otimes b_0)^{n_{a_0} \, \tilde{n}_{b_0}} \oplus (\text{other}) \neq 0 \ ,$$

as we have seen above that $a_0 \otimes b_0 \neq 0$, and by construction $n_{a_0}\, \tilde{n}_{b_0} > 0$. $\square$

The statement of the lemma is not true if $\mathcal{C}$ were just a finitely semisimple monoidal category. For example, take $A = \Bbbk \oplus \Bbbk$, A-mod with tensor product $\otimes_A$, the relative product over A. Write $1_1, 1_2$ for the units of the two summands $\Bbbk$ in A, so that the unit of A is $1_1 + 1_2$. Let $\Bbbk_1$ be the A-module where 1_1 acts as the identity and 1_2 acts as zero, and vice versa for $\Bbbk_2$. Then $\Bbbk_1$ and $\Bbbk_2$ are simple A-modules, but still we have

$$\Bbbk_1 \otimes_A \Bbbk_2 = 0 \ .$$

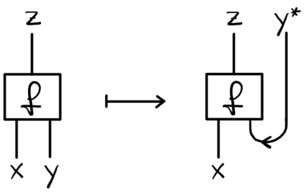
(Why does that hold? *Hint:* By definition of $\otimes_A$ we have, for all $v_i \in \Bbbk_i$ and $a \in A$, $(a.v_1) \otimes_A v_2 = v_1 \otimes_A (a.v_2)$.) (10.6.26)

■ As $\mathcal{C}$ is rigid, each object $X \in \mathcal{C}$ has a left dual X^* and a right dual ${}^*X$. It turns out that for fusion categories, these are isomorphic. We need the result of the following exercise:

Exercise 5.26. Let M be a rigid monoidal category. Show one of the following statements (not the one already shown in Exercise 2.22) and convince yourself that the others one can be seen similarly.

1. $M(x, z \otimes y^*) \cong M(x \otimes y, z) \cong M(y, {}^*x \otimes z)$, naturally in x, y, z.
2. $M(z \otimes {}^*y, x) \cong M(z, x \otimes y) \cong M(x^* \otimes z, y)$, naturally in x, y, z.

Hint: Use graphical calculus. The first isomorphism is the one from Exercise 2.22, for this one would use:



Do not forget to check naturality. What are the two functors for which the isomorphism are natural?

Exercise 5.27. Show that in the fusion category $\mathcal{C}$, the following statements about a nonzero object $X \in \mathcal{C}$ are equivalent: (What fails for $X = 0$?)

1. $\mathrm{End}(X) = \Bbbk\, 1_X$
2. X is simple,
3. X^* is simple,

4. ${}^*X$ is simple,

Hint: Show first that 1 and 2 are equivalent, then use Exercise 5.26.

Lemma 5.28. For all $X \in \mathcal{C}$ we have ${}^*X \cong X^*$.

Proof. It is enough to check the statement for X simple. (Why?) For X simple, consider the isomorphisms ?

$$\mathcal{C}(X,X) \overset{(1)}{\cong} \mathcal{C}(\mathbb{1}, X \otimes X^*) \overset{(2)}{\cong} \mathcal{C}(X \otimes X^*, \mathbb{1}) \overset{(3)}{\cong} \mathcal{C}(X^*, {}^*X)$$

Step 1 is Exercise 5.26 (1) with $x = \mathbb{1}$, $y = X$, $z = X$. Step 2 is Exercise 5.24 (4). Step 3 is Exercise 5.26 (1) with $x = X$, $y = X^*$, $z = \mathbb{1}$. As $\mathcal{C}(X,X)$ is nonzero, this shows that there is a nonzero morphism $X^* \to {}^*X$. By Exercise 5.27, both X^* and ${}^*X$ are simple. Hence they are isomorphic. $\square$

For info: The statement of the lemma does not hold in general rigid monoidal categories. There are examples of abelian rigid monoidal categories where the left and right dual of an object need not be isomorphic. For example, Hopf algebras for which the square of the antipode is not an inner automorphism. In [MaR, Sec. 4.3] one can find a $\mathbb{C}$-linear example where the left and right dual of a simple object are distinct.

5.4 Fusion rules and F-matrices

Let $\mathcal{C}$ be a fusion category, with $\Bbbk$ and $\mathcal{I}$ as in the previous section.

Definition 5.29. The *fusion rules* of $\mathcal{C}$ are the non-negative integers, for all $a, b, c \in \mathcal{I}$,

$$N_{ab}^{\ c} := \dim_{\Bbbk} \mathcal{C}(a \otimes b, c) \ .$$

Equivalently,

$$a \otimes b \cong \bigoplus_{c \in \mathcal{I}} c^{\oplus N_{ab}^{\ c}} \ .$$

■ We introduce a special notation for this direct sum decomposition. Namely, choose a basis

$$\lambda^{\alpha}_{(ab)c} : a \otimes b \to c \ , \quad \alpha = 1, \dots, N_{ab}^{\ c} \ ,$$

of $\mathcal{C}(a \otimes b, c)$. If $N_{ab}^{\ c} = 0$, no λ is chosen. If $a = \mathbb{1}$, then $\mathcal{C}(\mathbb{1} \otimes b, c) = 0$ for $b \neq c$, and for $b = c$ we have $\mathcal{C}(\mathbb{1} \otimes b, b) \cong \mathcal{C}(b,b) = \Bbbk\, 1_b$. Thus, $\mathcal{C}(\mathbb{1} \otimes b, b)$ is spanned by the unitor λ_b. Analogously, $\mathcal{C}(a \otimes \mathbb{1}, a)$ is spanned by ρ_a and we agree

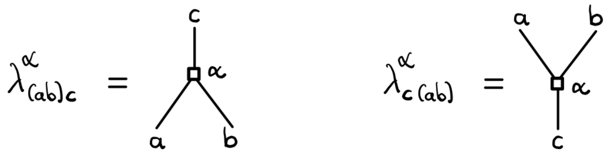
$$\lambda^{\bullet}_{(\mathbb{1}b)b} = \lambda_b : \mathbb{1} \otimes b \to b \quad , \quad \lambda^{\bullet}_{(a\mathbb{1})a} = \rho_a : a \otimes \mathbb{1} \to a \ .$$

Here, the dot "•" indicates that there is no multiplicity label in this case, i.e. the basis is indexed by the one-element set $\{\bullet\}$.

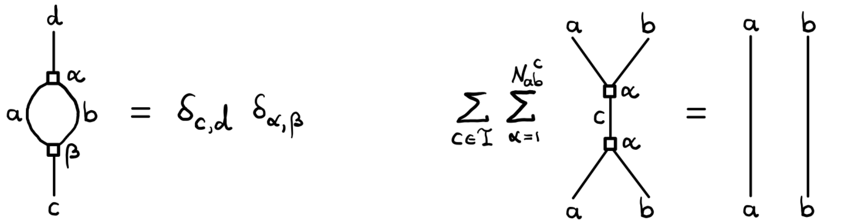
Denote by

$$\{\lambda^{\alpha}_{c(ab)}\}_{\alpha} \subset \mathcal{C}(c, a \otimes b)$$

the basis dual to $\{\lambda^{\alpha}_{(ab)c}\}_{\alpha}$ with respect to the non-degenerate pairing κ. We use the graphical notation



From Exercise 5.24 (4) we know that this dual basis pair exhibits $a \otimes b$ as the direct sum state in Definition 5.29. In graphical notation, this means



■ We can combine the observation that we can pull direct sums out of tensor products and Hom-spaces to get isomorphisms like the following:

$$\bigoplus_{q \in \mathcal{I}, \gamma=1,\dots,n_q} \mathcal{C}(q \otimes Y, Z) \to \mathcal{C}(X \otimes Y, Z) \ , \quad (f_{q,\gamma})_{q,\gamma} \mapsto \sum_{q \in \mathcal{I}} \sum_{\gamma=1}^{n_q} f_{q,\gamma} \circ (r_{q,\gamma} \otimes 1_Y) \ ,$$

where $r_{q,\gamma} : X \to q$ is the restriction map of the direct sum decomposition $X = \bigoplus_q q^{\oplus n_q}$. Let us apply this to $X = a \otimes b$, $Y = c$, $Z = d$, for $a, b, c, d \in \mathcal{I}$. We get the isomorphism

$$\begin{aligned} \bigoplus_{q \in \mathcal{I}, \gamma=1,\dots,N_{ab}^{\ q}} \mathcal{C}(q \otimes c, d) &\longrightarrow \mathcal{C}((a \otimes b) \otimes c, d) \\ (f_{q,\gamma})_{q,\gamma} &\longmapsto \sum_{q \in \mathcal{I}} \sum_{\gamma=1}^{N_{ab}^{\ q}} f_{q,\gamma} \circ (\lambda^{\gamma}_{(ab)q} \otimes 1_c) \ . \end{aligned}$$

As such, it maps a basis of the left hand side to a basis of the right hand side. For the left hand side we know a basis, namely $\{\lambda^{\delta}_{(qc)d}\}_{q,\gamma,\delta}$. On the right hand side we read off that

$$\big\{\lambda^{\delta}_{(qc)d} \circ (\lambda^{\gamma}_{(ab)q} \otimes 1_c)\big\}_{q,\gamma,\delta} \ \subset \ \mathcal{C}((a \otimes b) \otimes c, d)$$

is a basis, where $q \in \mathcal{I}$, $\gamma \in \{1, 2, \dots, N_{ab}^{\ q}\}$, $\delta \in \{1, 2, \dots, N_{qc}^{\ d}\}$. Analogously, we get the basis

$$\left\{\lambda^{\alpha}_{(ap)d} \circ (1_a \otimes \lambda^{\beta}_{(bc)p})\right\}_{p,\alpha,\beta} \subset \mathcal{C}(a \otimes (b \otimes c), d) \ .$$

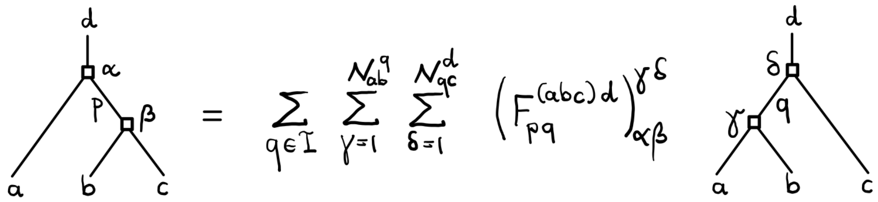
(Details? What are the ranges of the various indices?). By pre-composing the first basis above with the associator, we get another basis for $\mathcal{C}(a \otimes (b \otimes c), d)$. The F-matrices are the change-of-basis matrices: ?

Definition 5.30. The *F-matrices* of $\mathcal{C}$ with respect to the basis choices $\{\lambda^{\alpha}_{(ab)c}\}_{\alpha} \subset \mathcal{C}(a \otimes b, c)$ are defined by the identities

$$\lambda^{\alpha}_{(ap)d} \circ (1_a \otimes \lambda^{\beta}_{(bc)p}) \ = \ \sum_{q,\gamma,\delta} \left(F^{(abc)d}_{pq}\right)^{\gamma\delta}_{\alpha\beta} \lambda^{\delta}_{(qc)d} \circ (\lambda^{\gamma}_{(ab)q} \otimes 1_c) \circ \alpha_{a,b,c}$$

of morphisms in $\mathcal{C}(a \otimes (b \otimes c), d)$.

In graphical notation, this becomes



Example 5.31. Let $\omega \in Z^3(G, k^{\times})$ a normalised 3-cocycle and set $\mathcal{C} = \left(\mathsf{Vec}^{\omega}_{G}\right)^{\mathrm{fd}}$. As usual, we take $\Bbbk_g$ to be the one-dimensional vector space $\Bbbk$, concentrated in degree $g \in G$. We choose

$$\mathcal{I} = \left\{\Bbbk_g \,\middle|\, g \in G\right\} \ .$$

The fusion rules are

$$N_{gh}^{\ k} = \dim_{\Bbbk} \mathcal{C}(\Bbbk_g \otimes \Bbbk_h, \Bbbk_k) = \delta_{gh,k} \ .$$

(By our convention, we do not have $g \in \mathcal{I}$ but $\Bbbk_g \in \mathcal{I}$ and not g, we should really write $N_{\Bbbk_g \Bbbk_g}^{\ \Bbbk_h}$, but this is terrible, so we will not.) We have $N_{g,h}^{\ k} \in \{0, 1\}$, and so we can drop the multiplicity indices, we just have to be careful to keep track of when $N_{g,h}^{\ k} = 0$ to make sure the F-matrix entries are allowed by fusion. We define

$$\lambda_{(g,h)gh} : \Bbbk_g \otimes \Bbbk_h \to \Bbbk_{gh} \quad , \quad 1_g \otimes 1_h \mapsto 1_{gh} \ .$$

Then, for $a, b, c \in G$,

$$\lambda_{(a,bc)abc} \circ (1_a \otimes \lambda_{(b,c)bc})\big(1_a \otimes (1_b \otimes 1_c)\big) = 1_{abc}$$
$$\lambda_{(ab,c)abc} \circ (\lambda_{(a,b)ab} \otimes 1_c) \circ \alpha_{\mathbb{k}_a,\mathbb{k}_b,\mathbb{k}_c}\big(1_a \otimes (1_b \otimes 1_c)\big) = \omega(a,b,c)\, 1_{abc}$$

Comparing to Definition 5.30, we conclude

$$F^{(a,b,c)abc}_{bc,ab} = \omega(a,b,c)^{-1} \ .$$

Proposition 5.32. 1. The fusion rules satisfy $N_{a\mathbb{1}}^{\ b} = \delta_{a,b} = N^{b}_{\mathbb{1}a}$ and

$$\sum_{p\in\mathcal{I}} N_{bc}^{\ p} N_{ap}^{\ d} = \sum_{q\in\mathcal{I}} N_{ab}^{\ q} N_{qc}^{\ d} \ .$$

2. For each $a \in \mathcal{I}$ there is a unique $\bar{a}$ with $\bar{a} \cong a^* \cong {}^*a$. The map $a \mapsto \bar{a}$ is an involution on $\mathcal{I}$ such that $\bar{\mathbb{1}} = \mathbb{1}$ and $N_{ab}^{\ \bar{c}} = N_{\bar{b}\bar{a}}^{\ c}$ for all $a, b, c \in \mathcal{I}$.

3. If any of a, b, c are $\mathbb{1}$, the F-matrices take the form

$$\big(F^{(\mathbb{1}bc)d}_{db}\big)^{\bullet\delta}_{\bullet\beta} = \delta_{\beta,\delta} \ , \quad \big(F^{(a\mathbb{1}c)d}_{ca}\big)^{\bullet\delta}_{\alpha\bullet} = \delta_{\alpha,\delta} \ , \quad \big(F^{(ab\mathbb{1})d}_{bd}\big)^{\gamma\bullet}_{\alpha\bullet} = \delta_{\alpha,\gamma} \ .$$

4. Fix $a, b, c, d \in \mathcal{I}$. The matrix M with entries

$$M_{(p\alpha\beta),(q\gamma\delta)} = \big(F^{(abc)d}_{pq}\big)^{\gamma\delta}_{\alpha\beta}$$

is invertible. (What precisely is the index range of the matrix M? Check that part 1 guarantees that M is a square matrix.) Denote the inverse by ?

$$\big(G^{(abc)d}_{qp}\big)^{\alpha\beta}_{\gamma\delta} = (M^{-1})_{(q\beta\gamma),(p\alpha\beta)} \ .$$

5. The F-matrices satisfy the *pentagon identity*

$$\sum_{\nu} \big(F^{(abq)e}_{px}\big)^{\delta\nu}_{\alpha\beta} \big(F^{(xcd)e}_{qy}\big)^{\varepsilon\varphi}_{\nu\gamma} = \sum_{z,\zeta,\xi,\eta} \big(F^{(bcd)p}_{qz}\big)^{\zeta\xi}_{\beta\gamma} \big(F^{(azd)e}_{py}\big)^{\eta\varphi}_{\alpha\xi} \big(F^{(abc)y}_{zx}\big)^{\delta\varepsilon}_{\eta\zeta} \ .$$

Here, the sums are $\nu \in \{1, \dots, N_{xq}^{\ e}\}$ on the left hand side, and $z \in \mathcal{I}$ such that $N_{bc}^{\ z}, N_{zd}^{\ p}, N_{az}^{\ y} \neq 0$, and $\zeta \in \{1, \dots, N_{bc}^{\ z}\}$, $\xi \in \{1, \dots, N_{zd}^{\ p}\}$, $\eta \in \{1, \dots, N_{az}^{\ y}\}$ on the right hand side.

6. For all $a \in \mathcal{I}$ we have $F^{(a\bar{a}a)a}_{\mathbb{1}\mathbb{1}} = G^{(\bar{a}a\bar{a})\bar{a}}_{\mathbb{1}\mathbb{1}} \neq 0$.

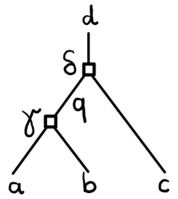
The proof will be partly an exercise and partly given after the following remark.

Remark 5.33. 1. Parts 1 and 2 of the proposition can be restated as follows. Let R be the free $\mathbb{Z}$-module spanned by $\mathcal{I}$, and denote the basis vectors by e_a, $a \in \mathcal{I}$. Set

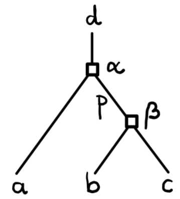
$$e_a \cdot e_b = \sum_{c\in\mathcal{I}} N_{ab}^{\;c}\, e_c \ .$$

Then R is a ring with unit $1 = e_{\mathbb{1}}$. Furthermore, define $\gamma : R \to R$ by $\gamma(e_a) = e_{\bar{a}}$. Then γ is an anti-involution of R: $\gamma^2 = \mathrm{id}$, $\gamma(1) = 1$ and $\gamma(r \cdot s) = \gamma(s) \cdot \gamma(r)$. This is the *fusion ring* of $\mathcal{C}$.

2. The matrix M in part 4 is precisely the change of basis used to define the F-matrix. The G-matrix is then the change of basis in the reverse direction:



$$= \sum_{p\in\mathcal{I}} \sum_{\alpha=1}^{N_{ap}^{d}} \sum_{\beta=1}^{N_{bc}^{p}} \left(G_{qp}^{(abc)d}\right)^{\alpha\beta}_{\gamma\delta}$$

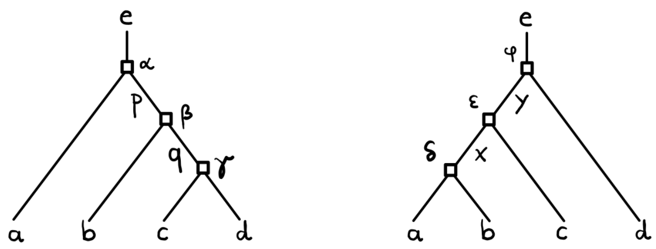


This implies

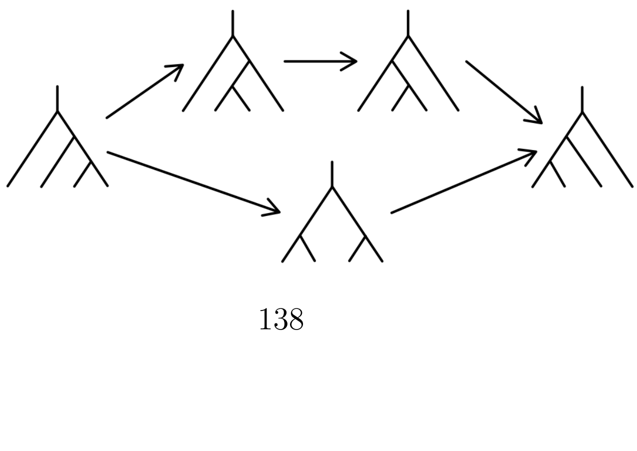
$$\sum_{q,\gamma,\delta} \big(F_{pq}^{(abc)d}\big)^{\gamma\delta}_{\alpha\beta} \big(G_{qp'}^{(abc)d}\big)^{\alpha'\beta'}_{\gamma\delta} \;=\; \delta_{p,p'}\, \delta_{\alpha,\alpha'}\, \delta_{\beta,\beta'} \ ,$$

$$\sum_{p,\alpha,\beta} \big(G_{qp}^{(abc)d}\big)^{\alpha\beta}_{\gamma\delta} \big(F_{pq'}^{(abc)d}\big)^{\gamma'\delta'}_{\alpha\beta} \;=\; \delta_{q,q'}\, \delta_{\gamma,\gamma'}\, \delta_{\delta,\delta'} \ .$$

3. The pentagon identity for the F-matrices follows by comparing two different routes between the following two bases of $\mathcal{C}(a\otimes(b\otimes(c\otimes d)), e)$



The two routes are



Proof of Proposition 5.32. Parts 1,2,3,5 are an exercise. Part 4 follows from part 2 of the above remark. It remains to show part 6.

Let $a \in \mathcal{I}$. From Lemma 5.28 we know that ${}^*a \cong a^*$, and by construction both are isomorphic to $\bar{a} \in \mathcal{I}$. Pick isomorphisms

$$L_a : a^* \to \bar{a} \quad , \quad R_a : {}^*a \to \bar{a} \ .$$

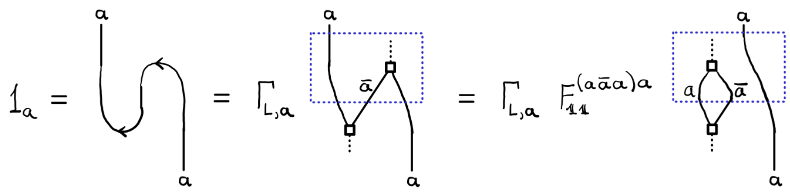
As the Hom-spaces involved are one-dimensional, by rescaling R and L appropriately, we may assume that

$$\begin{aligned} \mathrm{ev}_a &= \lambda^{\bullet}_{(\bar{a}a)\mathbb{1}} \circ (L_a \otimes 1_a) \ : \ a^* \otimes a \to \mathbb{1} \ , \\ \widetilde{\mathrm{ev}}_a &= \lambda^{\bullet}_{(a\bar{a})\mathbb{1}} \circ (1_a \otimes R_a) \ : \ a \otimes {}^*a \to \mathbb{1} \ . \end{aligned}$$

By the same token, there are constants $\Gamma_{L,a}, \Gamma_{R,a} \in \Bbbk^\times$ such that

$$\begin{aligned} \mathrm{coev}_a &= \Gamma_{L,a} \cdot (1_a \otimes L_a^{-1}) \circ \lambda^{\bullet}_{\mathbb{1}(a\bar{a})} \ : \ \mathbb{1} \to a \otimes a^* \ , \\ \widetilde{\mathrm{coev}}_a &= \Gamma_{R,a} \cdot (R_a^{-1} \otimes 1_a) \circ \lambda^{\bullet}_{\mathbb{1}(\bar{a}a)} \ : \ \mathbb{1} \to {}^*a \otimes a \ . \end{aligned}$$

To fix the coefficients, we check the four zig-zag identities. For example



and so $\Gamma_{L,a} F^{(a\bar{a}a)a}_{\mathbb{1}\mathbb{1}} = 1$ (there are no multiplicity indices here and we omit the $\bullet$). (On the right hand side, why does one not get a sum $\sum_q F^{(a\bar{a}a)a}_{\mathbb{1}q} \dots$?) ? The second zig-zag identity for the left dual gives $\Gamma_{L,a} G^{(\bar{a}a\bar{a})\bar{a}}_{\mathbb{1}\mathbb{1}} = 1$. This gives the condition in part 6. The same computation for the right duality produces

$$\Gamma_{R,a}\, G^{(a\bar{a}a)a}_{\mathbb{1}\mathbb{1}} = 1 \quad , \quad \Gamma_{R,a}\, F^{(\bar{a}a\bar{a})\bar{a}}_{\mathbb{1}\mathbb{1}} = 1 \ .$$

(Check one of these.) This is equivalent to the condition for the left duality (as both have to hold for all $a \in \mathcal{I}$). ? $\square$

Exercise 5.34. In this exercise, you prove parts of Proposition 5.32. E

1. Show parts 1 and 2. *Hint:* For part 2, show that in a rigid monoidal category one always has an isomorphism $X \to {}^*(X^*)$.

2. Show at least one of the identities in part 3.

3. Show part 5 – use the indexing as in part 2 of the above remark for easier comparison.

■ We now want to show the converse:

Proposition 5.35. Let

- $\mathcal{I}$ be a finite set with a distinguished element $\mathbb{1}$, and an involution $a \mapsto \bar{a}$ such that $\bar{\mathbb{1}} = \mathbb{1}$
- $N_{ab}^{\ c} \in \mathbb{Z}_{\geq 0}$ satisfy the relations in parts 1 and 2 of Proposition 5.32
- $\big(F_{pq}^{(abc)d}\big)_{\alpha\beta}^{\gamma\delta} \in \Bbbk$ satisfy the conditions in parts 3–6 of Proposition 5.32

Then there is a unique-up-to-equivalence fusion category $\mathcal{C}$ with these fusion rules and F-matrices.

The proof works by endowing $(\mathsf{Vec}_{\mathcal{I}})^{\mathrm{fd}}$ with a tensor product constructed from N and an associator obtained from F. The details are a bit lengthy and we give a sketch as an info point:

For info: *Proof.* Our starting point is the finitely semisimple category $\mathcal{C} := (\mathsf{Vec}_{\mathcal{I}})^{\mathrm{fd}}$ of finite-dimensional $\mathcal{I}$-graded vector spaces. As noted in Example 5.18 (1), we have $\mathcal{C} = A\text{-}\mathsf{mod}$ for $A = \Bbbk^{\oplus|\mathcal{I}|} = \bigoplus_{a\in\mathcal{I}} \Bbbk_a$, where $\Bbbk_a$ denotes the copy of $\Bbbk$ corresponding to $a \in \mathcal{I}$. We write $1_a \in \Bbbk_a$ for the copy of 1 in the a'th copy of $\Bbbk$. By construction, $\{1_a\}_{a\in\mathcal{I}}$ is a basis of A.

We describe the tensor product $\circledast$ on $\mathcal{C}$ via an A-$A\otimes A$-bimodule T. We take

$$T = \bigoplus_{a,b,c\in\mathcal{I}} t_{ab}^{\ c} \quad , \quad t_{ab}^{\ c} = \Bbbk^{\oplus N_{ab}^{\ c}} \ .$$

The right action of A on $x \in t_{ab}^{\ c}$ is $1_i.x = \delta_{i,c}x$, and the right action of $A\otimes A$ is $x.(1_i \otimes 1_j) = \delta_{i,a}\delta_{j,b}x$. For $M, N \in \mathcal{C}$ we define

$$M \circledast N := T \otimes_{A\otimes A} (M \otimes_\Bbbk N) \ ,$$

where on the right hand side we consider $M \otimes_\Bbbk N$ as an $A\otimes A$ left module.

We will abbreviate the n-fold tensor product of A with itself by A^n. The associator will be given as an A-A^3-bimodule morphism

$$\Phi : T \otimes_{A^2} (A \otimes_\Bbbk T) \longrightarrow T \otimes_{A^2} (T \otimes_\Bbbk A)$$

as

$$\alpha_{L,M,N} = \Big[L \circledast (M \circledast N) = T \otimes_{A^2} (L \otimes_\Bbbk (T \otimes_{A^2} (M \otimes_\Bbbk N)) \\ \xrightarrow{\sim} T \otimes_{A^2} (A \otimes_\Bbbk T) \otimes_{A^3} (L \otimes_\Bbbk M \otimes_\Bbbk N)$$

$$\xrightarrow{\Phi\otimes_{A^3}\mathrm{id}} T\otimes_{A^2}(T\otimes_{\Bbbk}A)\otimes_{A^3}(L\otimes_{\Bbbk}M\otimes_{\Bbbk}N)$$
$$\xrightarrow{\sim} T\otimes_{A^2}((T\otimes_{A^2}(L\otimes_{\Bbbk}M)\otimes_{\Bbbk}N) = (L\circledast M)\circledast N\Big] \ .$$

To construct Φ, we note

$$T\otimes_{A^2}(A\otimes_{\Bbbk}T)\cong\bigoplus_{a,b,c,d,p} t_{ap}^{\ d}\otimes_{\Bbbk} t_{bc}^{\ p}\ ,\quad T\otimes_{A^2}(T\otimes_{\Bbbk}A)\cong\bigoplus_{a,b,c,d,q} t_{qc}^{\ d}\otimes_{\Bbbk} t_{ab}^{\ q}\ .$$

To see this, consider

$$T\otimes_{\Bbbk}(A\otimes_{\Bbbk}T)\cong\bigoplus_{a,a',b,c,d,p,p'} t_{a'p'}^{\ d}\otimes_{\Bbbk}\Bbbk_a\otimes_{\Bbbk} t_{bc}^{\ p}\ .$$

The effect of the tensor product over A^2 is to set $a=a'$ and $p=p'$ as the balancing condition for $k_i\otimes k_j\in A^2$ forces $a=i=a'$ and $p=j=p'$. This explains the first isomorphism, the second is seen analogously.

Write $1^\alpha_{(ab)c}\in t_{ab}^{\ c}$ for the α'th copy of $\Bbbk$ in $t_{ab}^{\ c}$. The map Φ is defined as

$$\Phi\big(1^\alpha_{(ap)d}\otimes_{\Bbbk}1^\beta_{(bc)p}\big) \;=\; \sum_{q,\gamma,\delta}\big(G^{(abc)d}_{qp}\big)^{\alpha\beta}_{\gamma\delta}\ 1^\delta_{(qc)d}\otimes_{\Bbbk}1^\gamma_{(ab)q}\ .$$

Then α is invertible by property 4 in Proposition 5.32, $\Bbbk_{\mathbb{1}}$ is the tensor unit by property 1 and 3, and the pentagon identity for α translates into the pentagon identity for F, we skip the details.

To see that $\mathcal{C}$ as defined here reproduces the F-matrices it is constructed form, take

$$\lambda^\alpha_{(ab)c}: T\otimes_{A^2}(\Bbbk_a\otimes_{\Bbbk}\Bbbk_b)\to\Bbbk_c\ ,\quad 1^\beta_{(ab)d}\mapsto\delta_{c,d}\delta_{\alpha,\beta}1_c\ .$$

Comparing to Definition 5.30, we have

$$\lambda^\alpha_{(ap)d}\circ(\mathrm{id}_a\circledast\lambda^\beta_{(bc)p})\big(1^{\alpha'}_{(ap')d'}\otimes_{\Bbbk}1^{\beta'}_{(bc)p'}\big) = \delta_{d,d'}\delta_{p,p'}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}1_d$$
$$\lambda^\delta_{(qc)d}\circ(\lambda^\gamma_{(ab)q}\otimes 1_c)\circ\alpha_{a,b,c}\big(1^{\alpha'}_{(ap')d'}\otimes_{\Bbbk}1^{\beta'}_{(bc)p'}\big) = \delta_{d,d'}\big(G^{(abc)d}_{qp'}\big)^{\alpha'\beta'}_{\gamma\delta}\ 1_d\ ,$$

and indeed

$$\sum_{q,\gamma,\delta}\big(F^{(abc)d}_{pq}\big)^{\gamma\delta}_{\alpha\beta}\,\delta_{d,d'}\big(G^{(abc)d}_{qp'}\big)^{\alpha'\beta'}_{\gamma\delta}\ 1_d \;=\; \delta_{d,d'}\delta_{p,p'}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}1_d\ .$$

Finally, the left and right dual of $a\in\mathcal{I}$ is $\bar{a}$, and the duality maps are as in the proof of Proposition 5.32, we once more skip the details. □

The Fibonacci fusion category

We take $\Bbbk = \mathbb{C}$ and $\mathcal{I} = \{\mathbb{1}, \tau\}$ with fusion rules so that the fusion ring has $e_{\mathbb{1}}$ as unit and

$$e_\tau \cdot e_\tau = e_{\mathbb{1}} + e_\tau \ .$$

The involution is trivial, $\bar{\tau} = \tau$. As the fusion rules satisfy $N_{ab}^{\ c} \in \{0, 1\}$ we can omit all multiplicity indices. This makes the pentagon equation look much simpler,

$$F_{px}^{(abq)e} \, F_{qy}^{(xcd)e} \ = \ \sum_z F_{qz}^{(bcd)p} \, F_{py}^{(azd)e} \, F_{zx}^{(abc)y} \ .$$

Because of the unit conditions in Proposition 5.35 there are just five non-trivial F-matrix entries:

$$g := F_{\tau\tau}^{(\tau\tau\tau)\mathbb{1}} \quad , \quad f_{pq} := F_{pq}^{(\tau\tau\tau)\tau} \ , \quad p, q \in \mathcal{I} \ .$$

The pentagon equations have parameters $a, b, c, d, e, p, q, x, y \in \mathcal{I}$, which in our example a priori gives $2^9 = 512$ choices (not all are allowed by the fusion rules, of course), resulting in as many cubic equations for five unknowns.

Yet, we can write all solutions to these equations, subject to the conditions in Proposition 5.35. This does not take long, but is a bit tedious, so we collect the computation as an info-point.

For info: To reduce this number, one checks that the pentagon condition is non-trivial only if all four input objects a, b, c, d are different from $\mathbb{1}$. (Check this for one or two cases.) In our example, this means that $a = b = c = d = \tau$: ?

$$F_{px}^{(\tau\tau q)e} \, F_{qy}^{(x\tau\tau)e} \ = \ \sum_z F_{qz}^{(\tau\tau\tau)p} \, F_{py}^{(\tau z\tau)e} \, F_{zx}^{(\tau\tau\tau)y} \ .$$

Consider the following cases (we only list the values not forced by the fusion rules):

$$\begin{aligned} p = \mathbb{1}, y = \tau, x = \mathbb{1} : \ \ f_{\mathbb{1}\mathbb{1}} \ &= \ g \, f_{\mathbb{1}\tau} \, f_{\tau\mathbb{1}} \ . \\ p = \mathbb{1}, y = \tau, x = \tau : \ \ f_{\mathbb{1}\tau} \, f_{\tau\tau} \ &= \ g \, f_{\mathbb{1}\tau} \, f_{\tau\tau} \ . \end{aligned}$$

To get a rigid category, we need $f_{\mathbb{1}\mathbb{1}} \neq 0$. The first equation then implies $g, f_{\mathbb{1}\tau}, f_{\tau\mathbb{1}} \neq 0$. We can hence cancel in the second equation and conclude $g = 1$.

Setting $p = \mathbb{1}, y = \mathbb{1}$ results in $f_{\mathbb{1}\mathbb{1}} = f_{\mathbb{1}\tau} f_{\tau\mathbb{1}}$, which is already included above. Setting $p = \tau, y = \mathbb{1}$ equally gives no new condition. We can thus set $p = \tau, y = \tau$ in the general equation:

$$F_{\tau x}^{(\tau\tau q)e} \, F_{q\tau}^{(x\tau\tau)e} \ = \ \sum_z F_{qz}^{(\tau\tau\tau)p} \, F_{\tau\tau}^{(\tau z\tau)e} \, F_{zx}^{(\tau\tau\tau)\tau} \ .$$

Next consider the case

$$e = \mathbb{1}, q = \mathbb{1} \;:\; 1 \;=\; (f_{\mathbb{1}\mathbb{1}})^2 + f_{\mathbb{1}\tau}\, f_{\tau\mathbb{1}} \quad \Rightarrow \quad 1 \;=\; (f_{\mathbb{1}\mathbb{1}})^2 + f_{\mathbb{1}\mathbb{1}} \ .$$

Define

$$d_\pm = \frac{1 \pm \sqrt{5}}{2} \ .$$

Then $d_\pm$ gives the two solutions of $x^2 = x + 1$. Note that $d_\pm = 1 + (d_\pm)^{-1}$. Expressing the equation for $d_\pm$ in terms of $(d_\pm)^{-1}$ shows that $(d_\pm)^{-1}$ gives the two solutions of $y^2 + y = 1$. Hence we can set

$$f_{\mathbb{1}\mathbb{1}} = (d_\pm)^{-1} \ .$$

The remaining cases are

$$\begin{aligned}
&e = \mathbb{1}, q = \tau \;: && 1 \;=\; f_{\tau\mathbb{1}}\, f_{\mathbb{1}\tau} + (f_{\tau\tau})^2 \ . \\
&e = \tau, q = \mathbb{1}, x = \tau \;: && 1 \;=\; f_{\mathbb{1}\mathbb{1}} + (f_{\tau\tau})^2 \ . \\
&e = \tau, q = \tau, x = \mathbb{1} \;: && 1 \;=\; f_{\mathbb{1}\mathbb{1}} + (f_{\tau\tau})^2 \ . \\
&e = \tau, q = \tau, x = \tau \;: && (f_{\tau\tau})^2 \;=\; f_{\tau\mathbb{1}}\, f_{\mathbb{1}\tau} + (f_{\tau\tau})^3 \ .
\end{aligned}$$

The first three equations all state $(f_{\tau\tau})^2 = 1 - f_{\mathbb{1}\mathbb{1}}$. Substituting into the last equation gives $1 - f_{\mathbb{1}\mathbb{1}} = f_{\mathbb{1}\mathbb{1}} + f_{\tau\tau}(1 - f_{\mathbb{1}\mathbb{1}})$, which we can solve for $f_{\tau\tau}$:

$$f_{\tau\tau} = \frac{1 - 2f_{\mathbb{1}\mathbb{1}}}{1 - f_{\mathbb{1}\mathbb{1}}} = -(d_\pm)^{-1} \ .$$

This also solves $(f_{\tau\tau})^2 = 1 - f_{\mathbb{1}\mathbb{1}}$, and we are done.

All solutions to the pentagon equation compatible with the conditions in Proposition 5.35 are:

$$F^{(\tau\tau\tau)\mathbb{1}}_{\tau\tau} = 1 \ , \quad F^{(\tau\tau\tau)\tau}_{pq} = f_{pq} \quad , \quad \text{with} \quad \begin{pmatrix} f_{\mathbb{1}\mathbb{1}} & f_{\mathbb{1}\tau} \\ f_{\tau\mathbb{1}} & f_{\tau\tau} \end{pmatrix} = \begin{pmatrix} (d_\pm)^{-1} & w \\ (d_\pm w)^{-1} & -(d_\pm)^{-1} \end{pmatrix}$$

where

$$d_\pm = \frac{1 \pm \sqrt{5}}{2} \quad , \quad w \in \mathbb{C}^\times \text{ arbitrary} \ .$$

Note that d_+ is the golden ratio. (For info: one can check that different choices of w give equivalent fusion categories, but we will not do that here.)

Definition 5.36. $\mathsf{Fib}_\pm$ is the fusion category with the above fusion rules N and F-matrices as above for $d_\pm$ and $w = 1$. We write $\mathsf{Fib} = \mathsf{Fib}_+$.

Exercise 5.37. The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$. Show that for $n \geq 1$, E

$$\tau^{\otimes n} \;\cong\; \mathbb{1}^{\oplus F_{n-1}} \oplus \tau^{\oplus F_n} \ .$$

For info: The *rank* of a fusion category is the number of simple objects. Take $\Bbbk = \mathbb{C}$. There is just one fusion category of rank 1, namely $(\mathsf{Vec}_{\mathbb{C}})^{\mathrm{fd}}$. We have now actually met all fusion categories of rank 2. There are four: $(\mathsf{Vec}^{\omega}_{\mathbb{Z}_2})^{\mathrm{fd}}$ with $[\omega] \in H^3(\mathbb{Z}_2, \mathbb{C}^\times) \cong \mathbb{Z}_2$, and $\mathsf{Fib}_\pm$, see [Os1]. This is by no means an easy result: one has to show that there is no fusion category with fusion rule $e_\tau \cdot e_\tau = 1 + n e_\tau$ with $n \geq 2$. At the time of writing, fusion categories of rank 3 have not been classified! (But pivotal fusion categories of rank 3 have been classified in [Os2] and it is believed that this is also the answer for non-pivotal categories.)

The Ising fusion category

We take

$$\Bbbk = \mathbb{C} \ , \quad \mathcal{I} = \{\mathbb{1}, \sigma, \varepsilon\} \ . \tag{16.7.26}$$

The fusion ring is

$$e_\varepsilon \cdot e_\varepsilon = e_{\mathbb{1}} \ , \quad e_\sigma \cdot e_\varepsilon = e_\sigma \ , \quad e_\varepsilon \cdot e_\sigma = e_\sigma \ , \quad e_\sigma \cdot e_\sigma = e_{\mathbb{1}} + e_\varepsilon \ .$$

This example is already too big to easily solve the polynomial equations by hand. We will not list the F-matrices and instead refer to [DGNO, App. B] for the classification, and to [MuR, Sec. 4] for a collection of these results in the present notation. There are again two solutions which are characterised by the sign in $F^{(\sigma\sigma\sigma)\sigma}_{\mathbb{1}\mathbb{1}} = \pm 1/\sqrt{2}$. We refer to the Ising fusion category as $\mathsf{Is}_\pm$ and set $\mathsf{Is} = \mathsf{Is}_+$.

For info: The Ising fusion category is called that because it is equivalent as a $\mathbb{C}$-linear monoidal category to the category of representations of the irreducible Virasoro vertex operator algebra at central charge $c = \frac{1}{2}$. This Virasoro VOA in turn is part of the field content of the two-dimensional Ising conformal field theory, and so the fields of the Ising CFT organise into representations of (two copies – holomorphic and anti-holomorphic) this Virasoro VOA. For example, the primary energy field has weight $(\frac{1}{2}, \frac{1}{2})$ and is related to the object ε and the primary spin field has weights $(\frac{1}{16}, \frac{1}{16})$ and is related to σ.

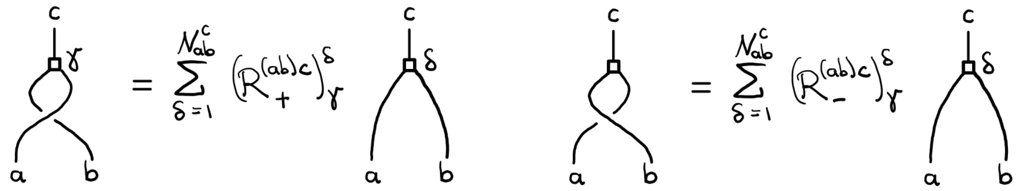
5.5 Ribbon fusion categories

A *ribbon fusion category* is simply a fusion category which is ribbon, i.e. which is braided and equipped with a ribbon twist.

■ Let $\mathcal{C}$ be a ribbon and let $\Bbbk$, $\mathcal{I}$, $\lambda^\alpha_{(ab)c}$ be as in the previous section. We can express the braiding and ribbon twist in a basis as we did for the associator an the F-matrix,

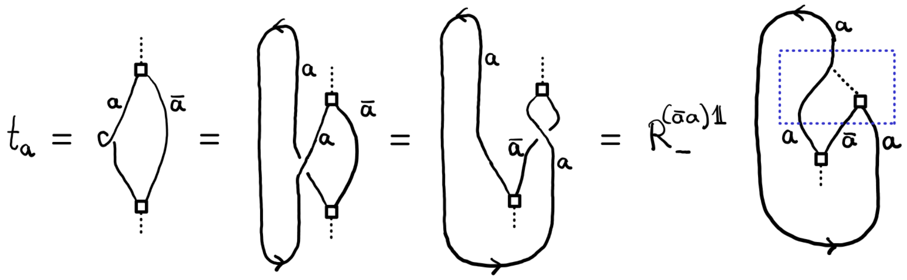
$$\lambda^\gamma_{(ba)c} \circ \beta^{\pm 1} = \sum_{\delta=1}^{N_{ab}^{\ c}} (R^{(ab)c}_{\pm})^\delta_\gamma \, \lambda^\delta_{(ab)c} \ : \ a \otimes b \to c \quad , \qquad \theta_a^{\pm 1} = t_a^{\pm 1} \, 1_a \ : \ a \to a \ .$$

In terms of string diagrams, the definition of R reads



The matrices $(R_\pm^{(ab)c})_\gamma^\delta$ are called the *R-matrices*. (For info: While that name is motivated by R-matrices in Hopf algebras, it is not the same. Still, from an R-matrix of a quasitriangular (finite-dimensional, semisimple) Hopf algebra one can compute an R-matrix as above.)

■ As an example, let us derive a formula for the dimension of $a \in \mathcal{I}$ in terms of F- and R-matrices and the t-values:



Inside the dotted box one can apply an F-move. This results in

$$\dim(a) = \frac{t_a}{R_-^{(\bar{a}a)\mathbb{1}} F_{\mathbb{1}\mathbb{1}}^{(a\bar{a}a)a}} \ .$$

Note that this shows in particular that in a ribbon fusion category

$$\dim(a) \neq 0 \quad \text{for all} \quad a \in \mathcal{I} \ ,$$

see also the comment to Exercise 5.24 (6) for general fusion categories.

Example 5.38. 1. *G-graded vector spaces:* Let G be a finite abelian group, q a quadratic form for G, and $\chi : G \to \{\pm 1\}$ a group homomorphism. Consider the ribbon fusion category $\mathcal{C} = \big(\mathsf{Vec}_G^{(\omega,\psi,\chi)}\big)^{\mathrm{fd}}$ from Example 5.5. We use the same notation as in Example 5.31. For the $t_g = t_{\Bbbk_g}$ we easily read off

$$t_g = \chi(g)\, q(g) \ .$$

For the R-matrix we compute

$$\lambda_{(h,g)hg} \circ \beta_{\Bbbk_g,\Bbbk_h}\big(1_g \otimes_\Bbbk 1_h\big) = \psi(g,h) 1_{hg} \quad , \quad \lambda_{(g,h)gh}\big(1_g \otimes_\Bbbk 1_h\big) = 1_{gh} \ .$$

As G is abelian we have $gh = hg$, and we read off

$$R_+^{(g,h)gh} = \psi(g,h) \ .$$

The above formula also gives the dimension (cf. Exercise 5.10 and Example 5.31, use that $R_-^{(\bar{a}a)\mathbb{1}} R_+^{(a\bar{a})\mathbb{1}} = 1$)

$$\dim \mathbb{k}_g = \chi(g)\, q(g)\, \psi(g, g^{-1})\, \omega(g, g^{-1}, g) \ .$$

2. *Fibonacci ribbon category:* We can equip each of Fib_ε, $\varepsilon = \pm$, with two ribbon structures, parametrised by $\nu = \pm$. Recall that there are no multiplicity labels. Set $h_{\mathbb{1}} = 0$ and $h_\tau = \nu \frac{2\pm\varepsilon}{5} \in \{\pm\frac{1}{5}, \pm\frac{3}{5}\}$. The ribbon structures corresponding to ν on Fib_ε is uniquely determined by

$$R_+^{(ab)c} = e^{\pi i (h_c - h_a - h_b)} \quad , \quad t_a = e^{2\pi i h_a} \ .$$

By the above formula, the dimensions are

$$\dim(\mathbb{1}) = 1 \ , \quad \dim(\tau) = d_\varepsilon = \frac{1}{2}\big(1 + \varepsilon\sqrt{5}\big) \ .$$

It would take us too far afield to check all the conditions (see the next info-point), we just refer to [DB].

3. *Ising ribbon category:* For the Ising fusion category Is we take

$$h_{\mathbb{1}} = 0 \ , \quad h_\sigma = \tfrac{1}{16} \ , \quad h_\varepsilon = \tfrac{1}{2} \ ,$$

and set, as above

$$R_+^{(ab)c} = e^{\pi i (h_c - h_a - h_b)} \quad , \quad t_a = e^{2\pi i h_a} \ .$$

This characterises a ribbon fusion category structure on Is. By the above formula, the dimensions are

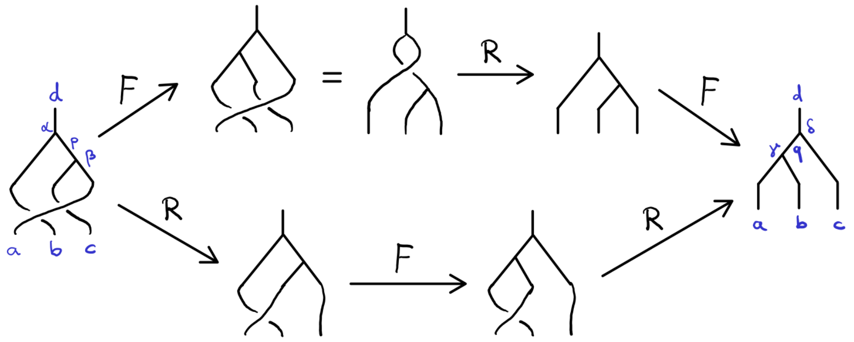
$$\dim(\mathbb{1}) = 1 \ , \quad \dim(\sigma) = \sqrt{2} \ , \quad \dim(\varepsilon) = 1 \ .$$

We again refer to [DGNO, App. B] and [MuR, Sec. 4] for details. (For info: It is shown in [DGNO, App. B] that are actually altogether 16 different ribbon fusion categories whose underlying fusion ring is that of the Ising fusion category.)

Exercise 5.39. Show that in a ribbon fusion category we have the identity Ⓔ

$$\sum_{x,\mu,\nu,\kappa} \big(F^{(bca)d}_{px}\big)^{\mu\nu}_{\alpha\beta} \big(R^{(ax)d}_{+}\big)^{\kappa}_{\nu} \big(F^{(abc)d}_{xq}\big)^{\gamma\delta}_{\kappa\mu} = \sum_{\rho,\sigma} \big(R^{(ac)p}_{+}\big)^{\rho}_{\beta} \big(F^{(bac)d}_{pq}\big)^{\sigma\delta}_{\alpha\rho} \big(R^{(ab)q}_{+}\big)^{\gamma}_{\sigma} \ .$$

Hint: Consider the transformations



One can again express the various defining conditions of a ribbon category in terms of F- and R-matrices and the t-values, but we will not go through this in detail.

For info: For the braiding, the statement is the following: Let $\mathcal{C}$ be a fusion category, with fixed set of representatives $\mathcal{I}$ and fixed choice of bases $\lambda^{\alpha}_{(ab)c}$. Let

$$(R^{(ab)c}_{\pm})^{\delta}_{\gamma} \in \Bbbk \quad , \quad a,b,c \in \mathcal{I},\ \gamma \in \{1,\dots,N_{ab}^{\ c}\},\ \delta \in \{1,\dots,N_{ab}^{\ c}\} \ .$$

The following are equivalent:

1. The morphisms, for $a,b \in \mathcal{I}$,

$$\beta_{a,b} = \sum_{c,\gamma,\delta} (R^{(ab)c}_{\pm})^{\delta}_{\gamma}\, \lambda^{\gamma}_{c(ba)} \circ \lambda^{\delta}_{(ab)c}$$

   extend to a (necessarily unique) braiding on $\mathcal{C}$.

2. For all $a,b,c \in \mathcal{I}$, R is invertible in the sense that

$$\sum_{\delta=1}^{N_{ba}^{\ c}} (R^{(ba)c}_{\pm})^{\delta}_{\gamma} (R^{(ab)c}_{\mp})^{\gamma'}_{\delta} = \delta_{\gamma,\gamma'} \ .$$

   Furthermore, the following equalities hold, for $\varepsilon = \pm$:

$$\sum_{x,\mu,\nu,\kappa} \big(F^{(bca)d}_{px}\big)^{\mu\nu}_{\alpha\beta} \big(R^{(ax)d}_{\varepsilon}\big)^{\kappa}_{\nu} \big(F^{(abc)d}_{xq}\big)^{\gamma\delta}_{\kappa\mu} = \sum_{\rho,\sigma} \big(R^{(ac)p}_{\varepsilon}\big)^{\rho}_{\beta} \big(F^{(bac)d}_{pq}\big)^{\sigma\delta}_{\alpha\rho} \big(R^{(ab)q}_{\varepsilon}\big)^{\gamma}_{\sigma}$$

Exercise 5.40. Let $\mathcal{C}$ be a braided fusion category with notation as in the above proposition. Let $t_a \in \Bbbk^\times$, $a \in \mathcal{I}$ be given. Show that $\theta_a = t_a\, 1_a$ extends to a balancing isomorphism on $\mathcal{C}$ if and only if (E)

$$t_c\, \delta_{\gamma,\gamma'} \;=\; t_a\, t_b \sum_\delta (R_+^{(ba)c})_\gamma^\delta\, (R_+^{(ab)c})_\delta^{\gamma'} \ .$$

for all a, b, γ, γ'.

- The *global dimension* of a ribbon fusion category $\mathcal{C}$ is

$$\mathrm{Dim}(\mathcal{C}) = \sum_{a\in\mathcal{I}} (\dim a)^2 \;\in\; \Bbbk \ .$$

Example 5.41. 1. *G-graded vector spaces:* For $\mathcal{C} = \big(\mathsf{Vec}_G^{(\omega,\psi,\chi)}\big)^{\mathrm{fd}}$, every object is tensor-invertible. By Exercise 5.17 (3) we have $\dim(\Bbbk_g)^2 = 1$ for every $g \in G$. Hence

$$\mathrm{Dim}(\mathcal{C}) = |G| \ .$$

If $|G| = 0$ in $\Bbbk$, this provides an example that the global dimension can be zero. (For info: For $\Bbbk$ algebraically closed and of characteristic zero, one can show that the global dimension of a ribbon fusion category is always nonzero, see [EGNO, Thm. 7.21.12].)

2. *Fibonacci ribbon category:*

$$\mathrm{Dim}(\mathsf{Fib}_\pm) = \dim(\mathbb{1})^2 + \dim(\tau)^2 = \frac{1}{2}(5 \pm \sqrt{5}) \ .$$

3. *Ising ribbon category:*

$$\mathrm{Dim}(\mathsf{Is}) = \dim(\mathbb{1})^2 + \dim(\sigma)^2 + \dim(\varepsilon)^2 = 1 + 2 + 1 = 4 \ .$$

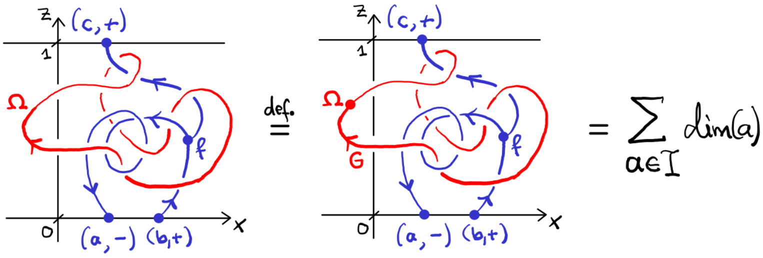
- The *minimal generator* of $\mathcal{C}$ is

$$G = \bigoplus_{a\in\mathcal{I}} a \ .$$

Here, "generator" means that for each $X \in \mathcal{C}$ there is n and a surjection (epimorphism) $G^{\oplus n} \to X$, and "minimal" means that it has the smallest number of direct summands. The *surgery colour* is

$$\Omega = \bigoplus_{a\in\mathcal{I}} \dim(a)\, 1_a \;:\; G \to G \ .$$

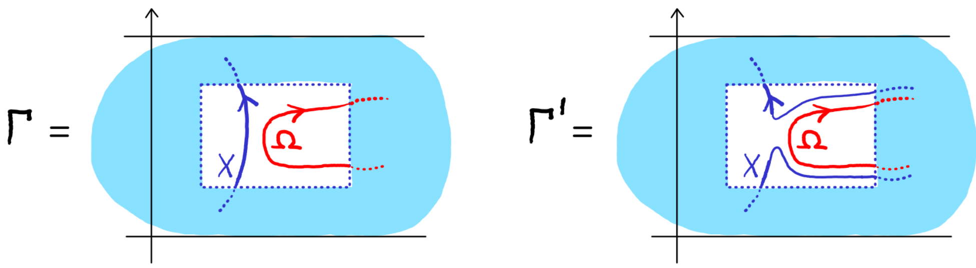
Let $\Gamma \in \mathrm{Diag}(\vec{a}, \vec{b})$ be a ribbon graph for the ribbon fusion category $\mathcal{C}$. We introduce a shorthand notation: A Ω-labelled annular ribbon that does not touch the boundary or a coupon (0-stratum) stands for an annular ribbon with G-labelled 1-stratum on which there is a unique 0-stratum, and this is labelled by Ω:



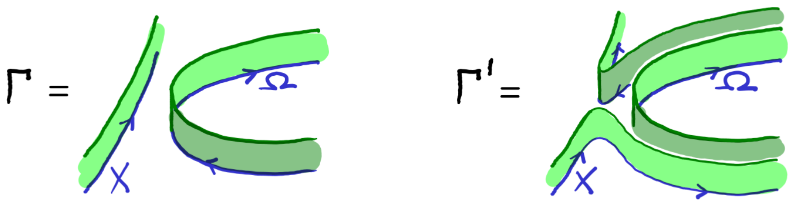
Here it is understood that $F_{\mathcal{C}}$ is applied to each ribbon graph.

Remark 5.42. The orientation of an Ω-labelled ribbon can be reversed without changing the value of $F_{\mathcal{C}}$. Indeed, combining the last expression with Exercise 5.16 shows that we can reverse the direction of the a-coloured ribbon at the same time as replacing a with $\bar{a}$ without changing the value of $F_{\mathcal{C}}$. By the same token, $\dim(a) = \dim(\bar{a})$, and as $a \mapsto \bar{a}$ is an involution (hence a bijection), the sum over a gives the same result.

Proposition 5.43. (Sliding property of the surgery colour) Let $\Gamma : \vec{a} \to \vec{b}$ be a ribbon graph with an annular ribbon labelled Ω. Let $\Gamma' : \vec{a} \to \vec{b}$ be obtained from Γ by "sliding a nearby ribbon over the annular ribbon Ω",



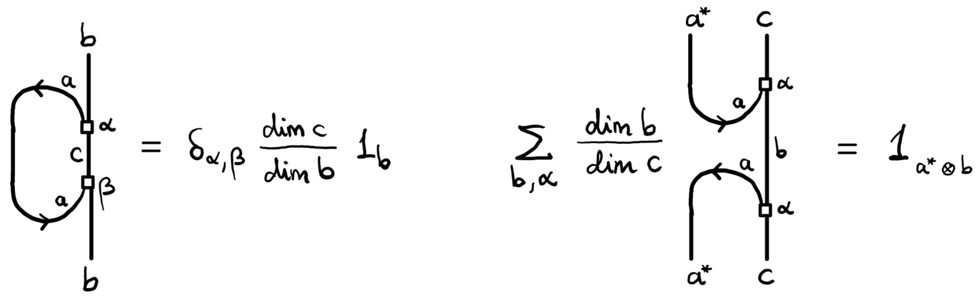
Here it is understood that the part of the ribbon graph in the blue shaded region is only changed near Ω as shown. In terms of ribbons, the change between Γ and Γ' locally looks as follows:



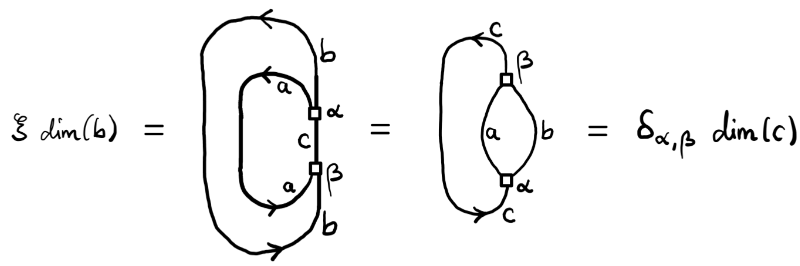
Then $F_{\mathcal{C}}(\Gamma) = F_{\mathcal{C}}(\Gamma')$. The same holds for the opposite orientation of the 1- and 2-strata of X.

The proof makes use of the following lemma:

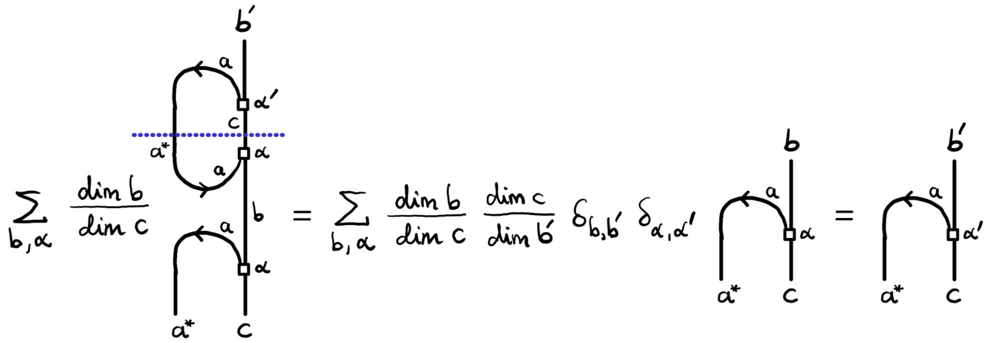
Lemma 5.44. We have



Proof. As b is simple, the left hand side of the first identity is equal to $\xi 1_b$ for some $\xi \in \Bbbk$. To compute ξ, we take the trace:

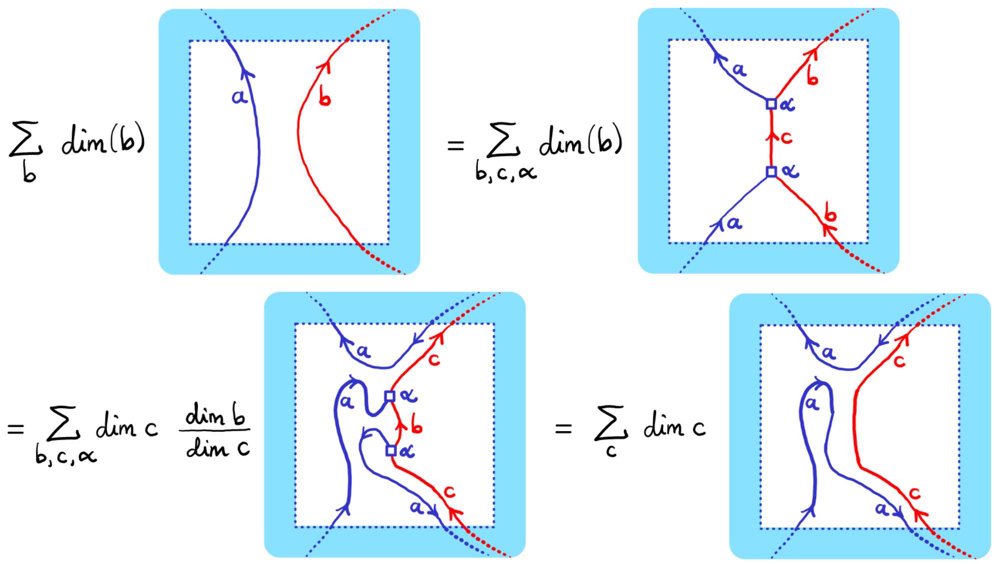


where the second equality is cyclicity of the trace. For the second claim of the lemma we use Exercise 5.24 (3): it is enough to check the identity after post-composing with a basis of $\mathcal{C}(a^* \otimes c, b)$. The first identity implies that the top half of the diagram on the left hand side is such a basis. We compute

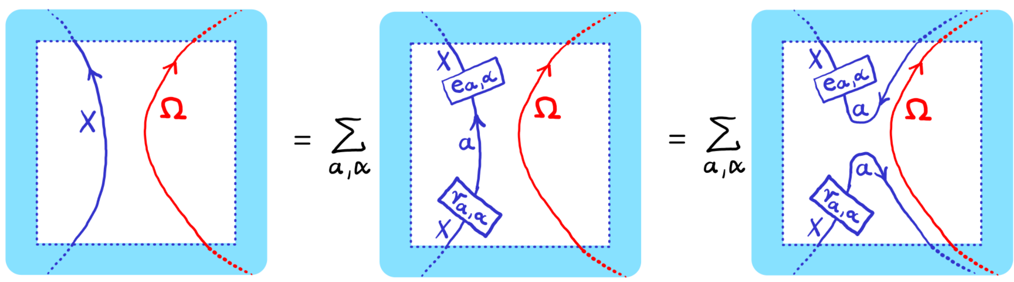


$\square$

Proof of Proposition 5.43. We first check the claim for $X = a$ simple. We only show the part of the ribbon graph in the dotted box, and it is understood that the functor $F_{\mathcal{C}}$ is applied to each diagram, that is, the equalities are of morphisms in $\mathcal{C}$, not of morphisms in $\mathsf{Rib}_{\mathcal{C}}$.

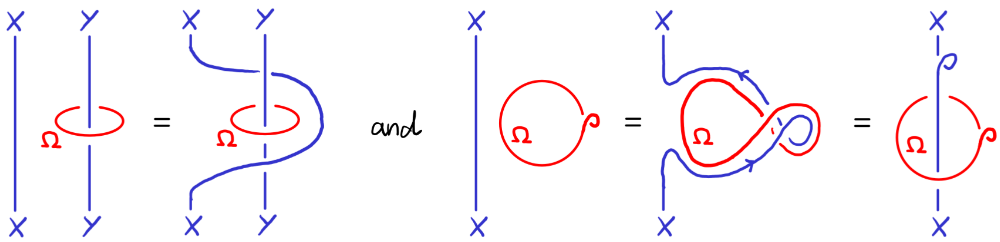


From this, we easily get the result for general X by expanding in simple objects,



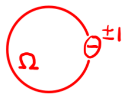
In the last expression, move $e_{a,\alpha}$ around the loop back in front of $r_{a,\alpha}$ and carry out the sum over a, α to get 1_X. $\square$

Remark 5.45. 1. Here are two examples of using the sliding property: (17.6.26)



2. An isolated Ω-loop evaluates to the global dimension $\mathrm{Dim}(\mathcal{C})$, and inserting a single ribbon twist $\theta^{\pm 1}$ results in constants $\Delta^{\pm} = \Delta^{\pm}(\mathcal{C})$ called *Gauss sums*:

$= \mathrm{Dim}\,\mathcal{C}$



$= \sum_a (\dim a)^2\, t_a^{\pm 1} =: \Delta_\pm$

Example 5.46. Let us compute the Gauss sums for the ribbon fusion categories in Examples 5.38 and 5.41:

1. *G-graded vector spaces:* For $\mathcal{C} = \big(\mathsf{Vec}_G^{(\omega,\psi,\chi)}\big)^{\mathrm{fd}}$, we get

$$\Delta_\pm = \sum_{g\in G} t_g^{\pm 1} \dim(\Bbbk_g)^2 = \sum_{g\in G} \chi(g)\, q(g)^{\pm 1} \ .$$

2. *Fibonacci ribbon category:* Let $\mathcal{C} = \mathsf{Fib}_\varepsilon$ with ribbon structure given by $\nu = \pm$. Write $\mathcal{D} = \sqrt{\mathrm{Dim}(\mathcal{C})}$ for the positive square root of $\mathrm{Dim}(\mathcal{C}) = \frac{1}{2}(5+\varepsilon\sqrt{5})$. Then

$$\Delta_\pm = 1 + t_\tau^{\pm 1} \dim(\tau)^2 = \mathcal{D}\, \exp(\mp\pi i\nu(3+4\varepsilon)/10) \ .$$

3. *Ising ribbon category:* For $\mathcal{C} = \mathsf{Is}$ we find

$$\Delta_\pm = \dim(\mathbb{1})^2 + t_\sigma^{\pm 1}\dim(\sigma)^2 + t_\varepsilon^{\pm 1}\dim(\varepsilon)^2 = 2\, e^{\pm\pi i/8} \ .$$

(For info: There is a relation to the central charge c in conformal field theory, given by $e^{2\pi i c/8} = \Delta_+/\mathcal{D}$. For example, the Ising CFT has $c = \frac{1}{2}$, which is consistent with $\mathcal{D} = 2$. For the Lee-Yang CFT one has $c = -\frac{22}{5}$, while the exponent above is, for $\varepsilon = \nu = -1$, $\exp(-\pi i/10)$ which gives $c = -\frac{2}{5} \mod 8$. So here one has to choose the negative square root for $\mathcal{D}$ to match the CFT result.)

Definition 5.47. Let R be a ribbon category. An object $T \in R$ is called *transparent* if

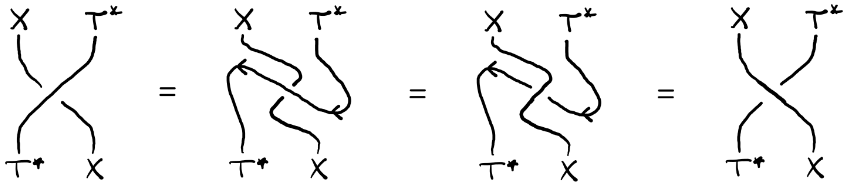
$$\beta_{X,T} \circ \beta_{T,X} = 1_{T\otimes X} \quad \text{for all } X \in R \ ,$$

or, equivalently,

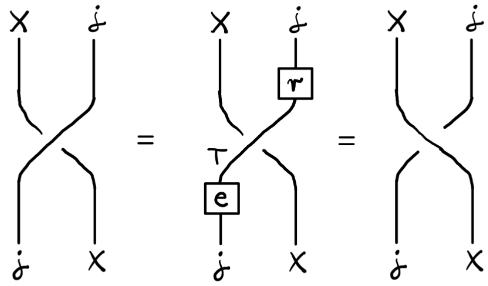
$$\beta_{T,X} = \beta_{X,T}^{-1} \quad \text{for all } X \in R \ .$$

The full subcategory of transparent objects is called the *symmetric centre* or *Müger centre* of R and is denoted by $Z_2(R)$.

Remark 5.48. 1. The symmetric centre is closed under tensor products and duals: If $S, T \in Z_2(R)$, so are $S \otimes T$ and T^*. For example,



$Z_2(R)$ is also closed under taking retracts. Indeed, if $T \in Z_2(R)$ and $r : T \leftrightarrow j : e$ is a retract, then $j \in Z_2(R)$:



2. By the previous point, the symmetric centre is hence a symmetric ribbon category. However, as opposed to the "canonical ribbon structure" we introduced on a rigid symmetric monoidal category (see Exercise 4.25), in general a symmetric ribbon category need *not* have trivial twist. This is because in a ribbon category, the expression for the right dual in terms of the left dual involves a braiding and a twist, see Section 5.1.

   However, the twist in a symmetric ribbon category always *squares* to one. (Why? *Hint:* Express the two twists in terms of braidings.) ?

   Take for example $\mathsf{Vec}_G^{(\omega,\psi,\chi)}$ with quadratic form $q = 1$, so that we can choose $\omega = 1$ and $\psi = 1$. Suppose there is a non-trivial group homomorphism $\chi : G \to \{\pm 1\}$ (e.g. for $G = \mathbb{Z}_2$). As $\psi(g,h) = 1$, the category is symmetric. But the t-values are $t_g = \chi(g)q(g) = \chi(g)$ and can be -1.

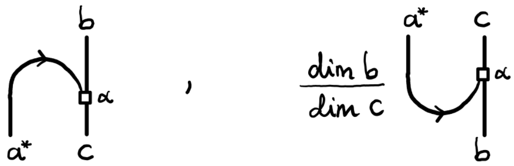
3. In a k-linear (or just additive) ribbon category, $Z_2(R)$ is also closed under taking direct sums. (Why?) ?

4. By parts 1 and 3, in a ribbon fusion category, all transparent objects are direct sums of simple transparent objects. In particular, the symmetric centre is still finitely semisimple. We write

$$\mathcal{I}_{\text{trans}} = \left\{ a \in \mathcal{I} \,\middle|\, a \text{ transparent} \right\} .$$

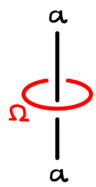
For info: Fusion categories are very hard to classify even over the complex numbers. However, surprisingly, *symmetric* fusion categories are classified by finite groups! In more detail, let G be a finite group, and let $z \in G$ be a central element such that $z^2 = 1$. Write $\mathrm{Rep}(G, z)$ for symmetric monoidal category of representations of G in $\mathsf{SVec}_{\Bbbk}^{\mathrm{fd}}$ such that z acts by parity (i.e. $z.v = (-1)^{|v|}v$). The following theorem is due to Deligne (see [EGNO, Thm. 9.9.26]): Let $\mathcal{C}$ be a symmetric fusion category over an algebraically closed field of characteristic zero. Then $\mathcal{C} \cong \mathrm{Rep}(G, z)$ as linear symmetric monoidal categories for some G, z. (The statement does not hold in finite characteristic, see [Os3] for extensions to this case.)

■ Recall from Lemma 5.44 the basis dual-basis pair



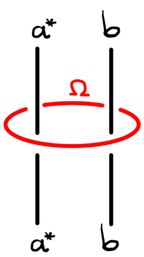
Lemma 5.49. We have

1.

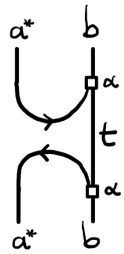


$$= \mathrm{Dim}\,\mathcal{C}\ \delta_{a \in \mathcal{I}_{\mathrm{trans}}}\ 1_a$$

2.



$$= \mathrm{Dim}\,\mathcal{C} \sum_{t \in \mathcal{I}_{\mathrm{trans}}} \sum_{\alpha} \frac{\dim t}{\dim b}$$

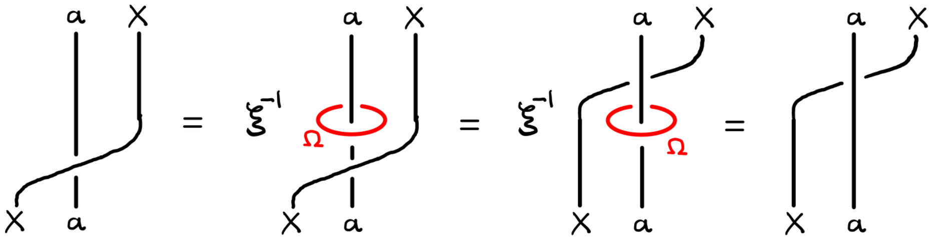


3.

$$\Delta_+(\mathcal{C})\Delta_-(\mathcal{C}) = \mathrm{Dim}(\mathcal{C}) \sum_{a \in \mathcal{I}_{\mathrm{trans}}} (\dim a)^2\, t_a = \mathrm{Dim}(\mathcal{C})\, \Delta_\pm(Z_2(\mathcal{C})) \ .$$

In the last equation we used that by Remark 5.48 (2) for symmetric ribbon fusion categories we have $\Delta_+(Z_2(\mathcal{C})) = \Delta_-(Z_2(\mathcal{C}))$.

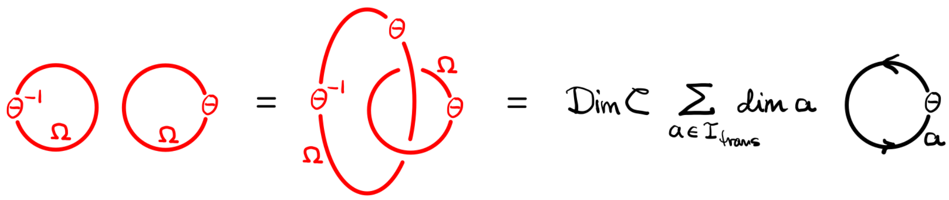
Proof. Part 1: Suppose $a \in \mathcal{I}_{\text{trans}}$. Then we can take a out of the Ω-loop to get $\mathrm{Dim}(\mathcal{C})1_a$, as required. Suppose $a \notin \mathcal{I}_{\text{trans}}$. As a is simple, the left hand side must be $\xi\, 1_a$ for some $\xi \in \Bbbk$. We need to show $\xi = 0$. Suppose $\xi \neq 0$. Then we can compute, for all $X \in \mathcal{C}$,



where in step 2 we used the sliding property similarly as in Remark 5.45 (1). Thus a is transparent, contradiction.

Part 2: we simply use the basis dual-basis pair above, substitute the second identity in Lemma 5.44 and use part 1 to restrict the sum from $\mathcal{I}$ to $\mathcal{I}_{\text{trans}}$. (Details?) ?

Part 3: We compute



The first step is the second example of the sliding property in Remark 5.45 (1), in the second step we cancel the twist and inverse twist and use part 1. $\square$

Corollary 5.50. For a symmetric ribbon fusion category $\mathcal{C}$, the Gauss sums satisfy

$$\Delta_\pm(\mathcal{C}) \in \{0, \mathrm{Dim}(\mathcal{C})\} \ .$$

Proof. If $\mathcal{C}$ is symmetric, then $Z_2(\mathcal{C}) = \mathcal{C}$. Let us write $\Delta(\mathcal{C}) = \Delta_\pm(\mathcal{C})$. Part 3 of the above lemma gives

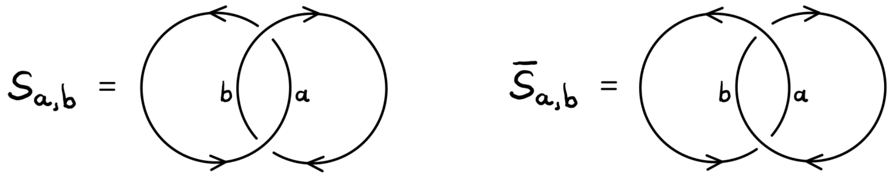
$$\Delta(\mathcal{C})^2 \ = \ \mathrm{Dim}(\mathcal{C})\, \Delta(\mathcal{C}) \ .$$

If $\Delta(\mathcal{C}) \neq 0$, we can cancel it and conclude $\Delta(\mathcal{C}) = \mathrm{Dim}(\mathcal{C})$. $\square$

■ We define the S-matrices of a ribbon fusion category $\mathcal{C}$ to be the $\mathcal{I} \times \mathcal{I}$-matrices (cf. Exercise 5.16 (2))

$$S_{a,b} = \mathrm{tr}_{a\otimes b}(\beta_{b,a} \circ \beta_{a,b}) \ , \quad \bar{S}_{a,b} = \mathrm{tr}_{a\otimes b}(\beta^{-1}_{a,b} \circ \beta^{-1}_{b,a}) \quad , \quad a, b \in \mathcal{I} \ .$$

As diagrams



By Exercise 5.16 (2) we have

$$S_{a,b} = S_{b,a} = S_{\bar{a},\bar{b}} \quad , \quad \bar{S}_{a,b} = S_{\bar{a},b} = S_{a,\bar{b}} \ .$$

Definition 5.51. A *modular fusion category* is a ribbon fusion category $\mathcal{C}$ for which the S-matrix is invertible.

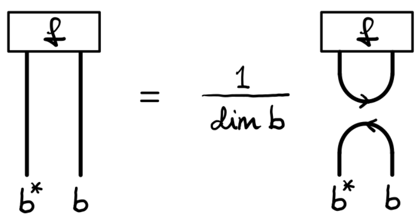
Exercise 5.52. Let $\mathcal{C}$ be a ribbon fusion category. Show that the following are equivalent: E

1. $\mathcal{C}$ is modular.

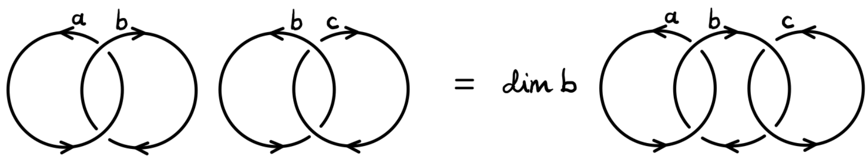
2. $\mathrm{Dim}(\mathcal{C}) \neq 0$ and every simple transparent object is isomorphic to the tensor unit.

Conclude that in a modular fusion category $\Delta_+(\mathcal{C})\Delta_-(\mathcal{C}) = \mathrm{Dim}(\mathcal{C}) \neq 0$.
Hint: For $1 \Rightarrow 2$ consider the columns of the S-matrix in case that $\mathcal{I}_{\text{trans}}$ contains a simple object a_0 other than $\mathbb{1}$. For $2 \Rightarrow 1$ show the following auxiliary statements:

a)



b)



c) $\sum_{b\in\mathcal{I}} S_{a,b}\, \bar{S}_{b,c} = \mathrm{Dim}(\mathcal{C})\, \delta_{a,c}$.

For info: The condition that $\mathrm{Dim}(\mathcal{C}) \neq 0$ in statement 2 can actually be dropped as it is implied by $Z_2(\mathcal{C}) \cong \mathsf{Vec}_{\Bbbk}^{\mathrm{fd}}$. This is not straightforward to prove, but follows when restricting the different characterisations of modularity in [Shi] to the case that the underlying category is semisimple.

Exercise 5.53. 1. Let $\mathcal{C}$ be a ribbon fusion category. Show that E

$$S_{a,b} = \frac{1}{t_a\, t_b} \sum_{c \in \mathcal{I}} N_{ab}^{\ c}\, t_c \, \dim(c) \ .$$

Hint: Use the result from Exercise 5.40.

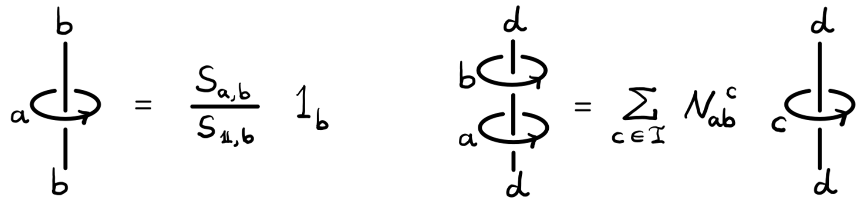
2. Let G be an abelian group such that $|G| \neq 0$ in $\Bbbk$. A quadratic form $q : G \to \Bbbk^{\times}$ is called *non-degenerate* if the associated bilinear form b (Definition 2.42) is non-degenerate. The pair (G, q) is then sometimes called a *metric group*. Show that $\left(\mathsf{Vec}_G^{(\omega,\psi,\chi)}\right)^{\mathrm{fd}}$ as in Example 5.5 is modular iff (G, q) is a metric group.

3. Fib_ε with ribbon structure ν, $\varepsilon, \nu = \pm$, and Is are modular. Show one of these statements.

Exercise 5.54. Let $\mathcal{C}$ be a modular fusion category. Prove the categorical Verlinde formula, E

$$N_{ab}^{\ c} = \frac{1}{\mathrm{Dim}(\mathcal{C})} \sum_{d \in \mathcal{I}} \frac{S_{ad}\, S_{bd}\, S_{\bar{c}d}}{S_{\mathbb{1}d}} \ .$$

(Note the bar in $S_{\bar{c}d}$.) *Hint:* First show the two identities:



Knot invariants

Given two k-linear categories C, D, we write $C \times_k D$ for the k-linear category whose objects are pairs (c, d) with $c \in C$ and $d \in D$, and whose morphism spaces are tensor products over k,

$$\mathrm{Hom}_{C \times_k D}((c, d), (c', d')) = C(c, c') \otimes_k D(d, d') \ .$$

By $C \boxtimes D$ we mean the category obtained from $C \times_k D$ after completing with respect to finite direct sums. (For info: The notation $C \boxtimes D$ can have many meanings. One which is prominently used is the *Deligne product of abelian categories*, see [EGNO, Sec. 1.11], which agrees with the notion given here if C and D are semisimple.)

If C, D are monoidal, braided, or ribbon, so are $C \times_k D$ and $C \boxtimes D$. If C, D are fusion, so is $C \boxtimes D$ (but not $C \times_k D$).

■ Let $\mathcal{C}$ be a ribbon fusion category over $\mathbb{C}$. Recall from Example 5.10 (2) the ribbon category $\mathcal{D} = \big(\mathsf{Vec}_G^{(1,\psi,1)}\big)^{\mathrm{fd}}$ of G-graded vector spaces (over $\mathbb{C}$), where we also took the grading group to be $G = \mathbb{C}$. Then $\mathcal{E} := \mathcal{C} \boxtimes \mathcal{D}$ is ribbon, but no longer fusion as it now has infinitely many simple objects,

$$(a, \Bbbk_z) \quad , \quad a \in \mathcal{I} \ , \ z \in \mathbb{C} \ .$$

Recall that the ribbon twist of $\Bbbk_z$ is $\exp(i\pi z^2)$. For $a \in \mathcal{I}$ pick $z_a \in \mathbb{C}$ such that $t_a = \exp(-i\pi z_a^2)$ and set $\tilde{a} = (a, \Bbbk_{z_a})$. Then by construction the twist of $\tilde{a}$ is trivial,

$$\theta_{\tilde{a}} = 1_{\tilde{a}} \ .$$

■ This allows one to define invariants of unframed oriented knots and links. Namely, let L be a link in $\mathbb{R}^3$. Add an arbitrary framing and label all components by $\tilde{a}$ as above (that is, make L into a ribbon-link in an arbitrary way, taking the 1-orientation of the link as that of the 1-stratum labelled by $\tilde{a}$). Call the resulting ribbon link $L_{\tilde{a}}^{\mathrm{rib}}$. Then

$$F_{\mathcal{E}}(L_{\tilde{a}}^{\mathrm{rib}})$$

is an invariant of L, independent of the choice of framing. Indeed, changing the framing will modify $F_{\mathcal{E}}(L_{\tilde{a}}^{\mathrm{rib}})$ by some power of $t_{\tilde{a}}$, which is one.

Exercise 5.55. The *HOMFLY polynomial* $P_L(v, z)$ is an invariant of an oriented (unframed) link L. It is a Laurent polynomial in v, z defined uniquely by $P_O(v, z) = 1$ (O is the unknot) and E

$$v^{-1} P_{L_+}(v, z) - v P_{L_-}(v, z) = z\, P_L(v, z) \ ,$$

where L_+, L_-, L are three links which only differ inside a 3-ball as in Exercise 5.17 (4). See e.g. [Mur, Def. 11.3.1] for more on the HOMFLY polynomial.

Carry out the above construction in the case $\mathcal{C} = \mathsf{Fib}_\pm$ with ribbon structure for $\nu = \pm$ as in Example 5.38 (2) and relate $F_{\mathcal{E}}(L_{\tilde{a}}^{\mathrm{rib}})$ to the HOMFLY polynomial $P_L(v, z)$, evaluated at a specific value for v and z. In more detail:

1. Show that the following identity holds in Fib_ε with ribbon structure $\nu = \pm$ and h_τ as in Example 5.38 (2): For $\xi = \exp(-\pi i h_\tau/2)$,

$$\xi^{-3}\, \beta_{\tau,\tau} + \xi^3\, \beta_{\tau,\tau}^{-1} = (\xi + \xi^{-1}) 1_{\tau\otimes\tau} \ .$$

2. Find $\tilde{\tau}$ and show that

$$\xi^{-7}\beta_{\tilde{\tau},\tilde{\tau}} + \xi^{7}\,\beta^{-1}_{\tilde{\tau},\tilde{\tau}} = (\xi + \xi^{-1})1_{\tilde{\tau}\otimes\tilde{\tau}} \ .$$

3. Show that the HOMFLY polynomial of two separate unknots (that is, the boundary of two embedded non-intersecting discs) is

$$P_{OO}(v,z) = (v^{-1} - v)/z \ .$$

Hint: This can be seen from a single application of the defining relation together with $P_O(v,z) = 1$ and recalling that $P_L(v,z)$ is an invariant of *unframed* oriented knots.

4. Given an explicit relation between $F_{\mathcal{E}}(L^{\mathrm{rib}}_{\tilde{a}})$ and $P_L(v,z)$. Check that your relation is compatible with the result in part 3. *Hint:* Check the formula

$$\dim(\tau) = \frac{\xi^7 + \xi^{-7}}{\xi + \xi^{-1}}$$

for all four choices $\varepsilon = \pm, \nu = \pm$.

5. Explain briefly why the expression in part 1 does not imply that $F_{\mathsf{Fib}_\varepsilon}$ itself is already an invariant of oriented links, even though it fits the expression of the HOMFLY polynomial.

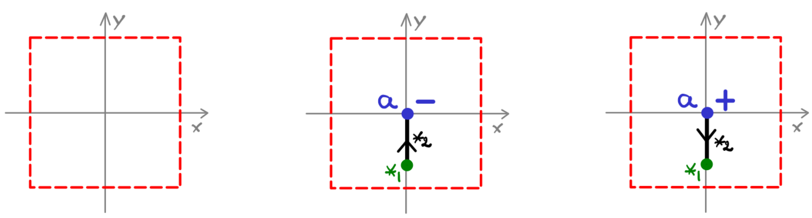
5.6 Kirby calculus and three-manifold invariants

Let $\mathcal{C}$ be a ribbon category – for now still general, not necessarily fusion or linear. Recall from Section 5.2 the label set $D(\mathcal{C}) = D_0 \sqcup D_1 \sqcup D_2 \sqcup D_3$, with

$$D_0 = \mathrm{mor}(\mathcal{C}) \sqcup \{*_0\} \ , \quad D_1 = \mathrm{obj}(\mathcal{C}) \sqcup \{*_1\} \ , \quad D_2 = \{*_2\} \ , \quad D_3 = \{*_3\} \ .$$

This was used to introduce 3-manifolds M with embedded $\mathcal{C}$-ribbon graph as $D(\mathcal{C})$-stratified 3-manifolds with very restricted possible stratifications (Definition 5.11). We now collect these into a suitable bordism category.

■ A $\mathcal{C}$-marked surface is a closed oriented surface Σ stratified by $D(\mathcal{C})_{\geq 1} = D_1 \sqcup D_2 \sqcup D_3$ (see above Definition 3.14) such that each point in Σ has a neighbourhood isomorphic to one of



The $\mathcal{C}$-labelled 0-strata are furthermore totally ordered, that is, labelled $1, \dots, n$. If we want to make the $\mathcal{C}$-labelling explicit we write $(\Sigma, \vec{a})$. Recall, also from above Definition 3.14, the notion of a stratified bordism.

Definition 5.56.

1. Let Σ_1, Σ_2 be $\mathcal{C}$-marked surfaces. A *bordism with embedded $\mathcal{C}$-ribbon graph*
$$M : \Sigma_1 \to \Sigma_2$$
is a $D(\mathcal{C})$-stratified bordism such that M is a 3-manifold with $\mathcal{C}$-ribbon graph.

2. The symmetric monoidal category
$$\mathsf{Bord}_3(\mathcal{C})$$
has $\mathcal{C}$-marked surfaces as objects and isomorphism classes of bordisms with embedded $\mathcal{C}$-ribbon graphs as morphisms.

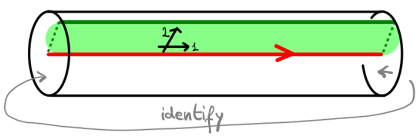
The orientation convention for boundary and boundary strata is as in Definition 5.11. This works out as the somewhat erratic $\psi_M : (\Sigma_1, -\vec{a}_1) \sqcup (-\Sigma_2, \vec{a}_2) \to \partial M$.

Remark 5.57. Given a ribbon graph $\Gamma \in \mathrm{Diag}(\vec{a}, \vec{b})$ we can identify $\mathbb{R}^2 \cup \{\infty\}$ with S^2 and use the embedding $\mathbb{R}^2 \hookrightarrow S^2$ to get a bordism $M_\Gamma : (S^2, \vec{a}) \to (S^2, \vec{b})$. This defines a functor
$$\mathsf{Rib}_{\mathcal{C}} \to \mathsf{Bord}_3(\mathcal{C}) \ .$$
However, this functor is not monoidal. (Why?)

■ Let $M : \Sigma_1 \to \Sigma_2$ be a bordism with embedded $\mathcal{C}$-ribbon graph Γ (by which we mean here $*_{0,1,2}$ and $\mathcal{C}$-labelled strata). By a *surgery link in M* we mean a ribbon link L in M, disjoint from Γ_M and ∂M, and coloured by Ω in the sense explained above Remark 5.42. The total ribbon graph in M with surgery link L is then $\Gamma_M \cup L$.

Definition 5.58. Given $M : \Sigma_1 \to \Sigma_2$ and a surgery link L in M, we define a new bordism $M_L : \Sigma_1 \to \Sigma_2$ as follows. For each component K of L:

1. Let T the stratified solid torus $D^2 \times S^1$, where D^2 is the two-dimensional disc, with stratification



Embed T in M via some $\varphi : T \to M$ such that the stratification above gets mapped to K, and such that $\varphi(T)$ is disjoint from the ribbon graph Γ in M, from ∂M, and from the other components of L.

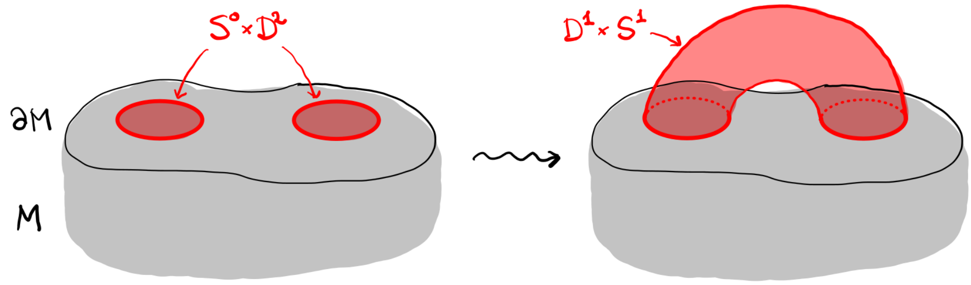
2. Remove the interior of $\varphi(T)$ and glue back the unstratified solid torus $T' = S^1 \times D^2$ via φ, restricted to the common boundary $S^1 \times S^1$ (and omit the leftover part of the stratification of $\varphi(T)$ on $\varphi(S^1 \times S^1)$).

We say that M_L *is obtained from* M *by surgery along* L.

Note that the surgery along L does not change M in a neighbourhood of the boundary ∂M and of the ribbon graph Γ_M. It thus makes sense to say

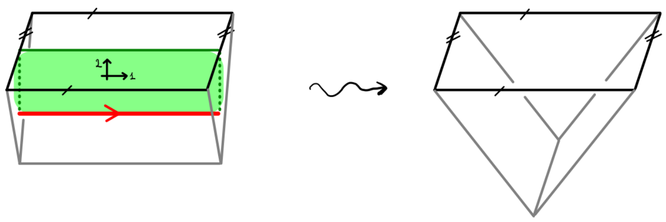
$$\partial M = \partial(M_L) \quad \text{and} \quad \Gamma_M = \Gamma_{M_L} \ .$$

For info: The best way to think about the above surgery operation is in terms of handle attachments to the 3-dimensional boundary of a 4-manifold. Consider the case of the 2-dimensional boundary of a 3-manifold first. An index 1 surgery on a 2-manifold is a replacement of $S^0 \times D^2$ with $D^1 \times S^1$, where $S^0 = \{+1, -1\}$ is the 0-sphere, and $D^1 = [-1, 1]$ is the 1-disc. Note that both pieces have the 1-dimensional boundary $S^0 \times S^1$. In terms of a 3-manifold with boundary, this amounts to attaching a 3-dimensional index 1 handle $D^1 \times D^2$,



Now back to 4-manifolds with 3-dimensional boundary. The surgery move in Definition 5.58 replaces $S^1 \times D^2$ with $D^2 \times S^1$ in the boundary. This amounts to attaching a 2-handle $D^2 \times D^2$. For more on this see [GS, Ch. 4.1].

Example 5.59. 1. One way to illustrate the surgery procedure is to draw the solid torus as a "piece of cake":



In these pictures, the gray rectangles are identified, as are the grey triangles. The back triangle then becomes the boundary 2-torus. The left picture is the solid torus shown in the above definition. The right picture shows the solid torus that gets glued in its place. Note that the boundary 2-torus is the same in both pictures.

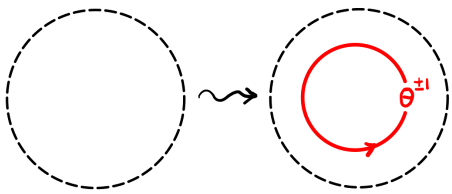
2. S^3 with surgery link L being an unknot (in paper plane framing) gives $(S^3)_L \cong S^2 \times S^1$. (Try to visualise this – with this example that is still possible.) ?

3. The three-ball B^3 as a bordism $B^3 : \emptyset \to S^2$, with surgery link L being $O_\pm$ (the unknot with an insertion of $\theta^{\pm 1}$) gives $(B^3)_L \cong B^3$. This example already shows that the surgery presentation will not be unique.

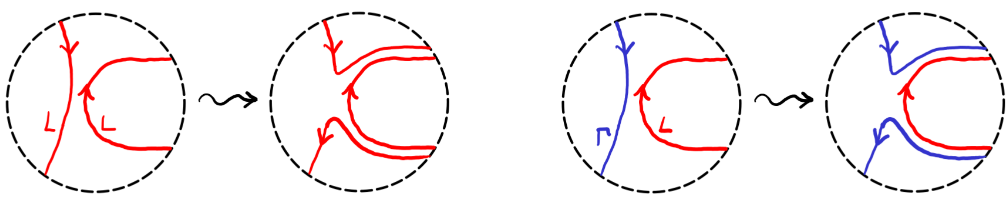
■ We need the following result from topology, see [Tu, Ch. II.2]:

Theorem 5.60. For $i = 1, 2$ let Γ_i be a ribbon graph in S^3 and L_i a surgery link in (S^3, Γ_i). Let $M_i = (S^3, \Gamma_i)_{L_i}$ be the 3-manifold with embedded ribbon graph obtained by surgery along L_i. Then M_1 and M_2 are isomorphic if and only if $\Gamma_1 \cup L_1$ and $\Gamma_2 \cup L_2$ are related by a finite sequence of the following moves:

1. A Kirby I move inside an embedded 3-ball (or its inverse):



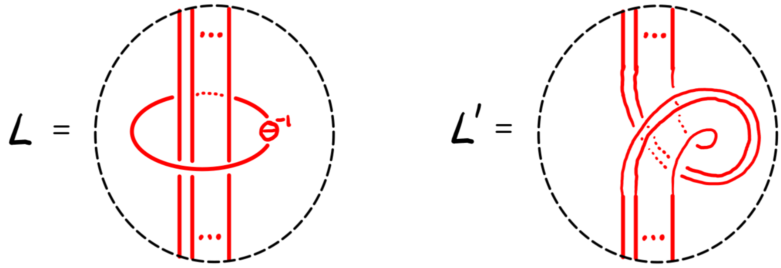
2. A Kirby II move for a component of L or for a component of Γ,



3. Orientation reversal on a component of L, and isotopies of Γ and L.

If we can assign a number to $(S^3, \Gamma \cup L)$ which does not change under the moves 1–3 above, we have obtained an invariant of closed, oriented 3-manifolds with embedded ribbon graphs.

Exercise 5.61. 1. Show that two surgery links L and L' in S^3 which in a 3-ball look like E



and which are identical outside the 3-ball are related by moves 1–3 and hence define the same 3-manifold, $(S^3)_L = (S^3)_{L'}$. The same holds when using the opposite twists/braidings for L, L' (but you do not need to also show this).

2. Consider S^3 with surgery link L being the Hopf link with paper-plane framing. Show that $(S^3)_L \cong S^3$. *Hint:* Use moves 1–3 to relate L to the empty knot: Consider the identity obtained by taking the trace of the identity in part 1 with $n = 2$. The use the identity for $n = 1$ in suitable places. E

■ From hereon,

$$\mathcal{C} \quad \text{is a ribbon fusion category} .$$

As a first step note that the ribbon graph $\Gamma \cup L$ in S^3 is always contained in some 3-ball, and we can use this to interpret it as a ribbon graph $\Gamma \cup L \in \mathrm{Diag}((),())$, i.e. with no ingoing or outgoing components. L is coloured by Ω, in the convention above Remark 5.42.

It then makes sense to apply the Reshetikhin-Turaev functor

$$F_{\mathcal{C}}(\Gamma \cup L) : \mathbb{1} \to \mathbb{1} .$$

We have:

Lemma 5.62. $F_{\mathcal{C}}(\Gamma \cup L)$ is invariant under moves 2 and 3 in Theorem 5.60.

Proof. The Kirby II moves are implied by the sliding property (Proposition 5.43). Isotopy invariance holds by Theorem 5.15, and invariance under orientation reversal of Ω-labelled components follows from $G \cong G^*$ and $\dim(a) = \dim(\bar{a})$. (Details?) □ ?

■ However, $F_{\mathcal{C}}(\Gamma \cup L)$ is not invariant under move 1. Instead we have

$$F_{\mathcal{C}}(\Gamma \cup L \cup O_\pm) = \Delta_\pm \, F_{\mathcal{C}}(\Gamma \cup L) ,$$

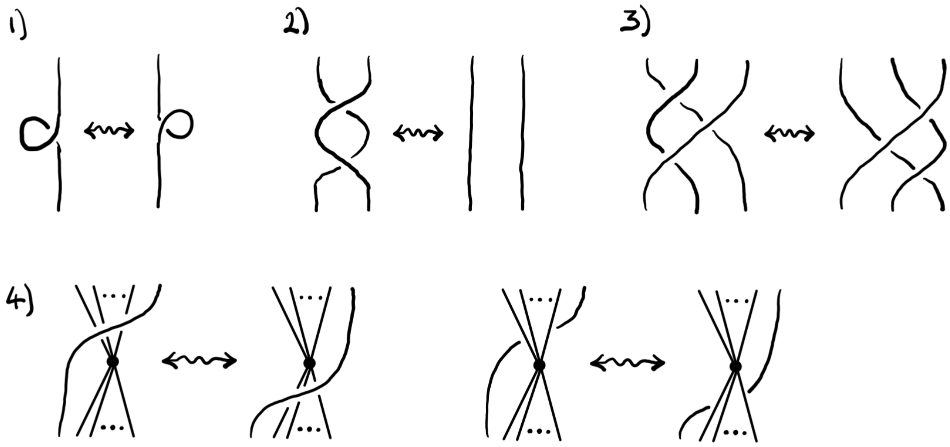


Figure 5.1: 1) framed Reidemeister I move; 2) Reidemeister II move; 3) Reidemeister III move; 4) Coupon IV move.

where $O_\pm$ stands for the unknot with an insertion of $\theta^{\pm 1}$, and we assume that $O_\pm$ is contained in a 3-ball which does not intersect the ribbon graph Γ and the remaining surgery link L.

To take care of this factor, we need the *signature of a link*.

Linking matrix and signature

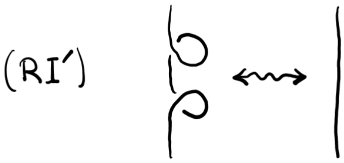
Let $\Gamma \in \mathrm{Diag}(\vec{a}, \vec{b})$ be a ribbon graph in $\mathbb{R} \times [0,1]$. By a *regular projection* of Γ we mean a projection of Γ to the x-z-plane so that we "end up with a graph in paper plane convention". That is, the ribbons are aligned in the y-direction and there are at most single crossings of 1-strata (the projection is injective on the $\mathcal{C}$-labelled part of the graph, except for possibly on isolated points on 1-strata, where double points are allowed, provided the two strands are transversal in the projection). In particular, twists of the ribbon need to be resolved in terms of crossings as on the left hand side of the diagram above Exercise 5.14.

A *planar isotopy* is an isotopy in $\mathbb{R} \times [0,1]$ of the graph obtained by the planar projection in which the crossings are treated as new 0-strata.

We import another theorem from topology (see [Tu, Ch. II.2]).

Theorem 5.63. Two ribbon graphs $\Gamma, \Gamma' \in \mathrm{Diag}(\vec{a}, \vec{b})$ are isotopic if and only if they are related by a finite sequence of planar isotopies and the moves in Figure 5.1. There, it is understood that the moves can be applied for arbitrary orientation of the 1-strata and arbitrary $\mathcal{C}$-labels.

Exercise 5.64. Show that in the above theorem one can replace the framed Reidemeister I move by the following:



That is, show that R I, R II, R III are equivalent to R I′, R II, R III.

■ Let $L = K_1 \cup \cdots \cup K_n$ be a ribbon link in $\mathbb{R}^3$ whose connected components we numbered $K_1, \dots, K_n$. Pick a regular projection of L. For each crossing c of L define the matrix $E_c \in \mathrm{Mat}_n(\mathbb{Z})$ as

where $E_{ij} \in \mathrm{Mat}_n(\mathbb{Z})$ are the standard unit matrices.

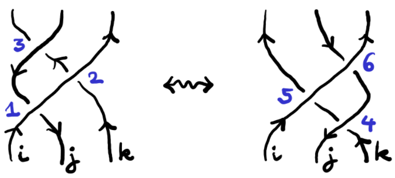
Definition 5.65. Let *linking matrix* $\mathrm{lk}(L) \in \mathrm{Mat}_n(\mathbb{Z})$ of a link L with ordered connected components $L = K_1 \cup \cdots \cup K_n$ is

$$\mathrm{lk}(L) = \sum_c E_c \ ,$$

where the sum is over all crossing of a regular projection of L.

Lemma 5.66. The linking matrix only depends on the isotopy class of L and the ordering of the components.

Proof. By Theorem 5.63 we need to show invariance under planar isotopies and the moves in Figure 5.1. Planar isotopies do not change the crossings of the projection, so that part is clear. Invariance under the Reidemeister moves follows from direct comparison, for example



The contributions of the six crossings $c_1, \dots, c_6$ are

$$E_{c_1} = -E_{ij} \ , \quad E_{c_2} = E_{ik} \ , \quad E_{c_3} = -E_{jk} \ ,$$

and

$$E_{c_4} = -E_{jk} \ , \quad E_{c_5} = E_{ik} \ , \quad E_{c_6} = -E_{ij} \ .$$

Thus indeed $E_{c_1} + E_{c_2} + E_{c_3} = E_{c_4} + E_{c_5} + E_{c_6}$. □

Example 5.67. 1. Let L be the Hopf link in paper plane framing, with the left circle being K_1 and the right circle K_2. There are two crossings, and one contributes E_{12} and the other E_{21}, so that

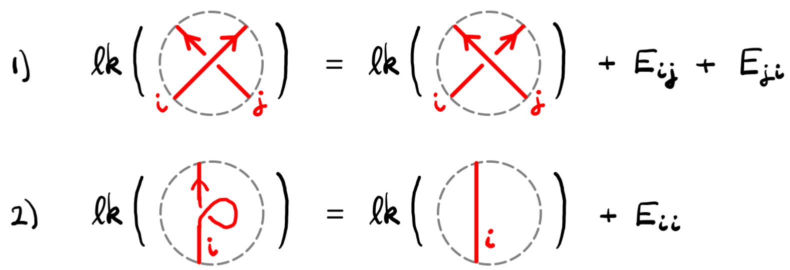
$$\mathrm{lk}(L) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

2. Let L be the unknot with an insertion of θ^n, $n \in \mathbb{Z}$. Then

$$\mathrm{lk}(L) = n .$$

To see this, consider the case $n > 0$. The n ribbon twists must be resolved in terms crossings as above Exercise 5.14. Each crossing contributes $E_c = E_{11}$, and so the sum is n. For $n < 0$ one obtaines $|n|$ times $E_c = -E_{11}$.

Lemma 5.68. The linking matrix changes as follows under modifications of the link inside a 3-ball as shown satisfies



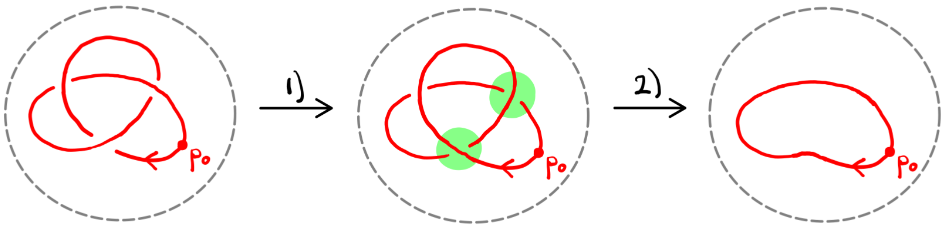
Furthermore, if L consists of n separated unknots O, the linking matrix is zero,

$$\mathrm{lk}(\underbrace{O \sqcup \cdots \sqcup O}_{n \text{ times}}) = 0 \in \mathrm{Mat}_n(\mathbb{Z}) .$$

These relations determine $\mathrm{lk}(L)$ uniquely.

Sketch of proof. It is easy to check that the relations are satisfied by $\mathrm{lk}(L)$. (Details?). To see that the relations determine $\mathrm{lk}(L)$, we show that they can be used to reduce L to n separated unknots. ?

For each component of K_i of L one carries out two steps: In step 1 we apply the first modification in Lemma 5.68 to achieve that all crossings of K_i with another component K_j ($j \neq i$) are such that K_i passes on top of K_j. After step 2, K_i is altogether on top of the remaining part of L and can be moved to the side. Pick a point on K_i and move in the direction of the orientation of K_i. Use the first modification in Lemma 5.68 so that the first time one passes a given crossing c one passes on top (and so the second time one passes, one necessarily passes below). For example,



Then the knot is "increasing" in the sense that the y-coordinate (pointing into the paper) can by thought of as increasing as one moves along K_i. But this means that K_i is an unknot with a certain number of twists, which can be removed by modification two. $\square$

Corollary 5.69. The linking matrix is symmetric.

Exercise 5.70. The linking matrix can also be computed from invariants of ribbon links obtained from a suitable ribbon category. Namely, let $\mathcal{C} = \big(\mathsf{Vec}_{\mathbb{C}}^{(1,\psi,1)}\big)^{\mathrm{fd}}$ be the ribbon category of $\mathbb{C}$-graded $\mathbb{C}$-vector spaces with $\psi(z,w) = e^{\pi i z w}$ as in Example 5.10 (2). Let $L = K_1 \cup \cdots \cup K_n$ be a ribbon link with connected components K_i. Pick $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and label K_i by $\mathbb{C}_{z_i}$. Call the resulting $\mathcal{C}$-labelled ribbon link Γ_z. Show that (E)

$$F_{\mathcal{C}}(\Gamma_z) \;=\; \exp\big(\pi i\, z^T \mathrm{lk}(L) z\big) \ ,$$

where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and z^T denotes the transpose. *Hint:* You can proceed as follows:

1. Consider the function $f : \mathbb{C}^n \to \mathbb{C}$, $f(z) = F_{\mathcal{C}}(\Gamma_z)$. Show that there is a unique symmetric matrix $M \in \mathrm{Mat}_n(\mathbb{Z})$ such that for all $z \in \mathbb{C}^n$,

$$f(z) = \exp\big(\,\pi i\, z^T M z\big) \ .$$

2. Show that M satisfies the relations in Lemma 5.68 and hence is equal to $\mathrm{lk}(L)$.

■ After these preparations, we can finally define the signature of a link. As the linking matrix is symmetric, it can be diagonalised over $\mathbb{R}$.

Definition 5.71. Let L be a link in $\mathbb{R}^3$. The *signature* $\mathrm{sig}(L)$ *of* L is the pair $(r,s) \in \mathbb{Z}_{\geq 0}$, where r is the number of positive eigenvalues of $\mathrm{lk}(L)$ and s the number of negative eigenvalues.

Example 5.72. For the linking matrices computed in Example 5.67, we get

1. For L be the Hopf link, the linking matrix has eigenvectors $(1, 1)$ and $(1, -1)$ of eigenvalues 1 and -1, and the signature is

$$\mathrm{sig}(L) = (1, 1) \ .$$

2. For L the unknot with θ^n, the linking matrix is a 1×1 matrix, hence already diagonal, and we find

$$\mathrm{sig}(L) = \begin{cases} (n, 0) & ; n > 0 \ , \\ (0, 0) & ; n = 0 \ , \\ (0, |n|) & ; n < 0 \ . \end{cases}$$

Lemma 5.73. Let L be a ribbon link in $\mathbb{R}^3$.

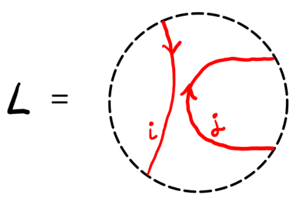
1. $\mathrm{sig}(L)$ is independent of the ordering of the components of L.

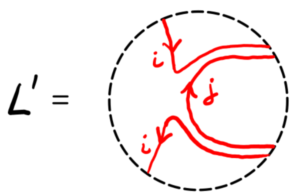
2. $\mathrm{sig}(L)$ is independent of the orientation of the components of L.

3. Let $O_\pm$ be a component of L which is an unknot with an insertion of $\theta^{\pm 1}$ and which is contained in a 3-ball containing no other parts of L. When removing $O_\pm$, the signature changes as

$$\mathrm{sig}(L \cup O_+) = \mathrm{sig}(L) + (1, 0) \quad , \quad \mathrm{sig}(L \cup O_-) = \mathrm{sig}(L) + (0, 1)$$

4. For $i \neq j$, the signature does not change when sliding component i of L over component j: We have $\mathrm{sig}(L) = \mathrm{sig}(L')$ with





Proof. Our main helper in this lemma is the real-valued instance of *Sylvester's law of inertia:*

> Let $M \in \mathrm{Mat}_n(\mathbb{R})$ be symmetric, and let $Q \in GL_n(\mathbb{R})$. For a real symmetric matrix X, let $n_{+,0,-}(X)$ be the number of positive/zero/negative eigenvalues of X. Then
>
> $$n_a(M) = n_a(Q^T M Q) \quad , \quad a = +, 0, - \ .$$

Parts 1 and 2 of the lemma are an exercise. For part 3, write O_m for the unknot with an insertion of θ^m, $m \in \mathbb{Z}$. Take O_m to be the last component of L. Then

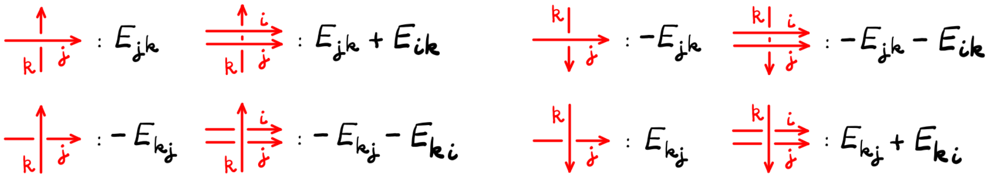
$$\mathrm{lk}(L \sqcup O_m) = \left(\begin{array}{c|c} \mathrm{lk}(L) & 0 \\ \hline 0 & m \end{array}\right)$$

and the signature changes as in Example 5.72 (2).

For part 3 one checks that

$$\mathrm{lk}(L') = (I + E_{ij})\,\mathrm{lk}(L)\,(I + E_{ji}) \ ,$$

where I is the $n \times n$ unit matrix. To see this, consider how the crossings of K_j with L relate before and after the sliding move. For $k \neq j$ (but possibly $k = i$) we get:



The case $k = j$ is an exercise. $\square$

Exercise 5.74. Prove parts 1 and 2 of the above lemma, as well as the missing case $k = j$ in part 4. (E)

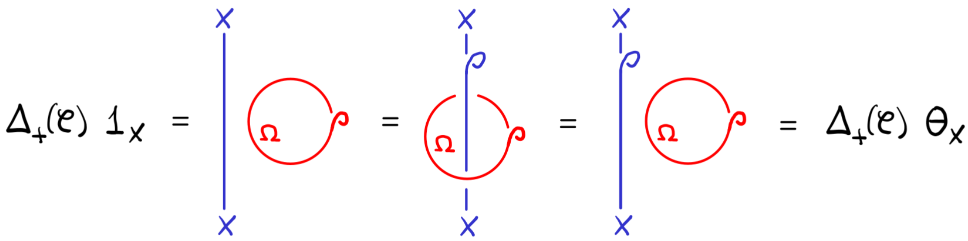
Witten-Reshetikhin-Turaev invariant

Definition 5.75. A ribbon fusion category $\mathcal{C}$ is *twist-nondegenerate* if $\Delta_+(\mathcal{C})$ and $\Delta_-(\mathcal{C})$ are nonzero.

In general, the symmetric centre of a ribbon fusion category may have non-trivial twists, see Remark 5.48 (2). For twist-nondegenerate categories this cannot happen:

Lemma 5.76. Let $\mathcal{C}$ be a twist-nondegenerate ribbon fusion category. Then for every $X \in Z_2(\mathcal{C})$ we have $\theta_X = 1_X$.

Proof. We compute



Step 2 is the sliding property as in Remark 5.45 (1) – this still holds for all $X \in \mathcal{C}$. In step 3 we use $X \in Z_2(\mathcal{C})$ to take it out of the Ω-coloured loop, that is, we use $\beta_{G,X} \circ \beta_{X,G} = 1_{X \otimes G}$. As $\Delta_+(\mathcal{C})$ is invertible, we conclude $1_X = \theta_X$. □

One immediate consequence is that for twist-nondegenerate ribbon fusion categories $\mathcal{C}$ one has

$$\Delta_\pm(Z_2(\mathcal{C})) = \mathrm{Dim}(Z_2(\mathcal{C})) \ ,$$

and hence also $\Delta_+(\mathcal{C})\Delta_-(\mathcal{C}) = \mathrm{Dim}(\mathcal{C})\,\mathrm{Dim}(Z_2(\mathcal{C}))$, cf. Lemma 5.49 (3). In particular,

$$\mathrm{Dim}(\mathcal{C}) \neq 0 \ , \quad \mathrm{Dim}(Z_2(\mathcal{C})) \neq 0 \ .$$

■ Let M be a closed connected 3-manifold with embedded $\mathcal{C}$-ribbon graph, and let L be a surgery link in (S^3, Γ) such that

$$M \ \cong \ (S^3, \Gamma)_L \ .$$

The *Witten-Reshetikhin-Turaev invariant*, or *surgery invariant*, of M for a twist-nondegenerate ribbon fusion category $\mathcal{C}$ is defined to be

$$\mathrm{Inv}_{\mathcal{C}}(M) \ = \ \frac{F_{\mathcal{C}}(\Gamma \cup L)}{\Delta_+^r \, \Delta_-^s} \ .$$

Here, $(r,s) = \mathrm{sig}(L)$, and $F_{\mathcal{C}}(\Gamma \cup L)$ is understood as above Lemma 5.62: The ribbon graph $\Gamma \cup L$ is seen as an element of $\mathrm{Diag}((),())$ and L is coloured by Ω. If $M = M_1 \sqcup \cdots \sqcup M_n$ with M_i connected, we define $\mathrm{Inv}_{\mathcal{C}}(M)$ to be the product of the connected components,

$$\mathrm{Inv}_{\mathcal{C}}(M_1 \sqcup \cdots \sqcup M_n) \ = \ \mathrm{Inv}_{\mathcal{C}}(M_1) \cdots \mathrm{Inv}_{\mathcal{C}}(M_n) \ .$$

The main result of this section is: (24.6.26)

Theorem 5.77. Let $\mathcal{C}$ be a twist-nondegenerate ribbon fusion category. Then $\mathrm{Inv}_{\mathcal{C}}(M)$ only depends on the isomorphism class of M.

Proof. We may assume that M is connected. By Theorem 5.60, it is enough to show that $\mathrm{Inv}_{\mathcal{C}}(M)$ is invariant under moves 1–3 listed there.

Move 1: Below Lemma 5.62 we saw that $F_{\mathcal{C}}(\Gamma \cup L \cup O_\pm) = \Delta_\pm \, F_{\mathcal{C}}(\Gamma \cup L)$. By Lemma 5.73 (3), the signature behaves as $\mathrm{sig}(L \cup O_+) = \mathrm{sig}(L) + (1,0)$ and $\mathrm{sig}(L \cup O_-) = \mathrm{sig}(L) + (0,1)$. Combining this shows that $\mathrm{Inv}_{\mathcal{C}}(M)$ is invariant under the Kirby-I move.

Move 2 and 3: By Lemma 5.62, $F_{\mathcal{C}}(\Gamma \cup L)$ does not change under moves 2 and 3. By Lemma 5.73 (2,4), also the signature does not change. □

Example 5.78. The invariant of S^3 with empty ribbon graph is simply

$$\mathrm{Inv}_{\mathcal{C}}(S^3) = 1 \ ,$$

as the signature of the empty link is $(0, 0)$ and $F_{\mathcal{C}}$ of the empty ribbon graph is 1. In Exercise 5.61 we saw that also the paper plane framed Hopf link L is a surgery representation of S^3. Let us compute $\mathrm{Inv}_{\mathcal{C}}(S^3)$ in that presentation. By Example 5.72 (1), the signature of the Hopf link is $(1, 1)$. The value of $F_{\mathcal{C}}$ is

$$\begin{aligned} F_{\mathcal{C}}(L) &\overset{(1)}{=} \sum_{a\in\mathcal{I}} \dim(a)\, F_{\mathcal{C}}(L_{a,\Omega}) \overset{(2)}{=} \sum_{a\in\mathcal{I}} \dim(a)^2 \ \mathrm{Dim}(\mathcal{C})\, \delta_{a\in\mathcal{I}_{\mathrm{trans}}} \\ &\overset{(3)}{=} \mathrm{Dim}(\mathcal{C}) \ \mathrm{Dim}(Z_2(\mathcal{C})) \overset{(4)}{=} \Delta_+(\mathcal{C})\, \Delta_-(\mathcal{C}) \ . \end{aligned}$$

In step 1 we expand one of the two Ω-coloured components of L into a sum over simple objects. Accordingly, $L_{a,\Omega}$ stands for the Hopf link with one component coloured a and the other Ω. In step 2 we apply Lemma 5.49 (1), step 3 is just the definition of the global dimension of $Z_2(\mathcal{C})$, and step 4 is the observation below Definition 5.75. Altogether, as it has to, also this presentation gives $\mathrm{Inv}_{\mathcal{C}}(S^3) = 1$.

Exercise 5.79. Let $\mathcal{C}$ be a twist-nondegenerate ribbon fusion category.

1. Let $M = S^2 \times S^1$ with a ribbon labelled $X \in \mathcal{C}$ placed along the S^1-direction. Write $X = X_0 \oplus X_1$, where X_0 is the maximal transparent subobject of X. Show that

   $$\mathrm{Inv}_{\mathcal{C}}(M) = \mathrm{Dim}(\mathcal{C}) \ \dim(X_0) \ .$$

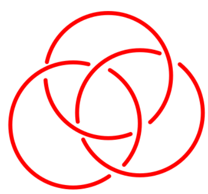
   You may use that $M \cong (S^3, \Gamma)_L$ with $\Gamma \cup L$ being the Hopf link (in paper plane framing), one component of which is coloured X, and the other component is the surgery link L. In particular, for $X = \mathbb{1}$ we get

   $$\mathrm{Inv}_{\mathcal{C}}(S^2 \times S^1) = \mathrm{Dim}(\mathcal{C}) \ .$$

2. Suppose in addition that $\mathcal{C}$ is modular. Show that

   $$\mathrm{Inv}_{\mathcal{C}}(T^3) = \mathrm{Dim}(\mathcal{C})^2 \, |\mathcal{I}| \ .$$

   Here, T^3 is the 3-torus, and $|\mathcal{I}|$ denotes the number of elements in $\mathcal{I}$. You may use that the 3-torus as a surgery presentation $T^3 \cong (S^3)_L$ with L being the *Borromean Link*:

Hint: Expand one of the Ω-colours as a sum over simple objects. Use Lemma 5.49 (2). Check that $\mathrm{lk}(L) = 0 \in \mathrm{Mat}_3(\mathbb{Z})$.

■ Suppose that $\mathcal{C}$ is a twist-nondegenerate *symmetric* ribbon category. Let M be a closed oriented 3-manifold with embedded ribbon graph Γ. We claim that

$$\mathrm{Inv}_{\mathcal{C}}(M) = \mathrm{Dim}(\mathcal{C})^{b_1(M)} L(M) \ .$$

Here,

- $b_1(M)$ is the first Betti number of M, that is, the rank of the abelian group $H_1(M, \mathbb{Z})$, or, equivalently, $b_1 = \dim_{\mathbb{Q}}(H_1(M, \mathbb{Q}))$. Recall the notation $n_{+,0,-}$ for the eigenvalues of the linking matrix of a link J. As an input from topology we will use that if $M \cong (S^3)_J$, then

$$b_1(M) = n_0 = |J| - n_+ - n_- \ ,$$

  where $|J|$ is the number of components of J and (n_+, n_-) is the signature of J. This follows from [GS, Prop. 5.3.11].

- $L(M)$ is the trivial TFT with line defects from Section 4.4 in the case of 3 dimensions. As L expects only 0 and 1 strata, when writing $L(M)$ one only keeps the $\mathcal{C}$-labelled strata in M and forgets the $*_{0,1,2}$-labelled strata (that is, relabels them by $*_3$). One also implicitly applies the conversion between 0-strata with only out-going 1-strata as used for L and the convention with in- and outgoing 1-strata used here.

  Note that L was constructed from a rigid symmetric monoidal category, which can be seen as a ribbon category with trivial twist (cf. Exercise 4.25). In the present setting it follows from twist-nondegeneracy that $\mathcal{C}$ has trivial twist (Lemma 5.76).

Exercise 5.80. Show the above claim. *Hint:* By Lemma 5.76, $\mathcal{C} = Z_2(\mathcal{C})$ has trivial twist and $\mathrm{Dim}(\mathcal{C}) = \Delta_\pm(\mathcal{C})$. You may use the slogan "evaluating L on (M, Γ) amounts to forgetting the manifold and evaluating the string diagram for Γ".

5.7 The universal construction and 3d TFT

The universal construction allows one to turn invariants of manifolds, seen as bordisms $\emptyset \to \emptyset$, into functors out of the corresponding bordism category. Here we discuss a variation of that construction. We fix:

- $\mathcal{C}$: a ribbon fusion category over an algebraically closed field $\Bbbk$ (of arbitrary characteristic), with representative set $\mathcal{I}$ for its simple objects.

- $\mathcal{C}$ is assumed to be twist-nondegenerate. We pick

$$\mathcal{D} \in \Bbbk \quad , \quad \text{such that} \;\; \mathcal{D}^2 = \Delta_+(\mathcal{C})\Delta_-(\mathcal{C}) \, (\neq 0) \ .$$

- $\mathcal{S} := Z_2(\mathcal{C})$ is the symmetric centre of $\mathcal{C}$, its simple objects have representatives $\mathcal{I}_{\text{trans}}$.

Our aim is to build a 3d TFT

$$\tau_{\mathcal{C}} : \mathsf{Bord}_3(\mathcal{C}) \to \mathcal{S} \ .$$

Note that if $\mathcal{C}$ is in addition modular, then $\mathcal{S} = \mathsf{Vec}^{\text{fd}}$ and so we get a 3d TFT valued in vector spaces.

Step 1: Normalisation of the invariant

We would like to start from an invariant $\tau_{\mathcal{C}}(M)$ of closed 3-manifolds with embedded ribbon graph which has a chance of being the value of a TFT on bordisms $\emptyset \to \emptyset$. Proposition 4.17 gives a constraint in that case: For $\Sigma \in \mathsf{Bord}_3(\mathcal{C})$ we must have

$$\tau_{\mathcal{C}}(\Sigma \times S^1) = \dim(\tau_{\mathcal{C}}(\Sigma)) \ ,$$

where on the right hand side we have the categorical dimension in $\mathcal{S}$.

We start by noting that $\mathrm{Inv}_{\mathcal{C}}(M)$ from the previous section does not directly have that property. For example:

- For $\mathcal{C}$ modular, $\mathcal{S} = \mathsf{Vec}^{\text{fd}}$ and $\tau_{\mathcal{C}}(\Sigma \times S^1)$ must be a non-negative integer. In Exercise 5.79 we saw

$$\mathrm{Inv}_{\mathcal{C}}(S^2 \times S^1) = \mathrm{Dim}(\mathcal{C}) \ , \quad \mathrm{Inv}_{\mathcal{C}}(T^2 \times S^1) = \mathrm{Dim}(\mathcal{C})^2 \, |\mathcal{I}| \ ,$$

  which are in general not integers.

- For $\mathcal{C}$ symmetric we saw in Exercise 5.80 that $\mathrm{Inv}_{\mathcal{C}}(M) = \mathrm{Dim}(\mathcal{C})^{b_1(M)} L(M)$. Since L is a TFT, we get

$$\mathrm{Inv}_{\mathcal{C}}(\Sigma \times S^1) = \mathrm{Dim}(\mathcal{C})^{b_1(\Sigma \times S^1)} \dim(L(\Sigma)) \ .$$

  Again we are off by some power of $\mathrm{Dim}(\mathcal{C})$.

This motivates the following definition. Recall that $\mathcal{D}$ is a choice of square root of $\Delta_+(\mathcal{C})\Delta_-(\mathcal{C})$. Set

$$\delta := \frac{\Delta_+}{\mathcal{D}} \quad , \quad \sigma(L) = r - s \text{ for } (r,s) = \mathrm{sig}(L) \ .$$

Definition 5.81. Let M be connected and let $(S^3, \Gamma)_L$ be a surgery representation of M. The *normalised Witten-Reshetikhin-Turaev*, or *normalised surgery invariant*, of M is

$$\tau_{\mathcal{C}}(M) := \frac{\mathcal{D}}{\mathrm{Dim}(\mathcal{C})} \mathcal{D}^{-|L|} \delta^{-\sigma(L)} F_{\mathcal{C}}(\Gamma \cup L) \ .$$

For non-connected M, $\tau_{\mathcal{C}}(M)$ is defined to be the product of $\tau_{\mathcal{C}}$ applied to the connected components of M.

Since now $\Delta_+ = \delta\,\mathcal{D}$ and $\Delta_- = \mathcal{D}/\delta$, for M connected we get

$$\begin{aligned}\mathrm{Inv}_{\mathcal{C}}(M) &= \frac{F_{\mathcal{C}}(\Gamma \cup L)}{\Delta_+^r\,\Delta_-^s} = \mathcal{D}^{-r-s}\,\delta^{-r+s}\,F_{\mathcal{C}}(\Gamma \cup L) \\ &= \mathcal{D}^{n_0}\,\mathcal{D}^{-|L|}\,\delta^{-\sigma(L)}\,F_{\mathcal{C}}(\Gamma \cup L) = \frac{\mathrm{Dim}(\mathcal{C})}{\mathcal{D}}\,\mathcal{D}^{b_1(M)}\,\tau_{\mathcal{C}}(M) \ .\end{aligned}$$

Here we used $|L| = r + s + n_0$ with n_0 the zero eigenvalues of $\mathrm{lk}(L)$, as well as $n_0 = b_1(M)$. In particular, $\tau_{\mathcal{C}}(M)$ is again independent of the choice of surgery presentation.

Example 5.82. 1. For $\mathcal{C}$ modular we have

$$\mathcal{D}^2 \ = \ \Delta_+(\mathcal{C})\,\Delta_-(\mathcal{C}) \ = \ \mathrm{Dim}(\mathcal{C})\ \mathrm{Dim}(Z_2(\mathcal{C})) \ = \ \mathrm{Dim}(\mathcal{C}) \ .$$

Using $b_1(S^2 \times S^1) = 1$ and $b_1(T^2 \times S^1) = 3$, this gives

$$\tau_{\mathcal{C}}(S^2 \times S^1) = 1 \ , \quad \tau_{\mathcal{C}}(T^2 \times S^1) = |\mathcal{I}| \ .$$

2. For $\mathcal{C}$ symmetric, the same computation for $\mathcal{D}^2$ gives

$$\mathcal{D}^2 \ = \ \mathrm{Dim}(\mathcal{C})^2 \ ,$$

and we make the choice $\mathcal{D} = \mathrm{Dim}(\mathcal{C})$. In Exercise 5.79 we saw $\mathrm{Inv}_{\mathcal{C}}(S^2 \times S^1) = \mathrm{Dim}(\mathcal{C})$ also for $\mathcal{C}$ not modular, and so we again get $\tau_{\mathcal{C}}(S^2 \times S^1) = 1$. The statement $\mathrm{Inv}_{\mathcal{C}}(M) = \mathrm{Dim}(\mathcal{C})^{b_1(M)} L(M)$ from Exercise 5.80 turns into

$$\tau_{\mathcal{C}}(M) = L(M) \ .$$

In particular, all 3-manifolds with empty ribbon graph evaluate to 1:

$$L(S^2 \times S^1) = 1 \ , \quad L(T^2 \times S^1) = 1 \ .$$

Since also very surface without marked points evaluates to $\mathbb{1}$, this is consistent: $L(S^2) = \mathbb{1}$, $L(T^2) = \mathbb{1}$.

Step 2: Restriction to anomaly-free case

It will be central in the discussion below that we can trade complicated topology for a more complicated embedded ribbon graph without changing the value of the invariant $\tau_{\mathcal{C}}$.

Namely, let $L_1 \cup L_2$ be a surgery link, which we collect in two parts L_1, L_2 (which may be linked). Then we would like to compare

$$\tau_{\mathcal{C}}\big((S^3,\Gamma)_{L_1\cup L_2}\big) \quad \text{and} \quad \tau_{\mathcal{C}}\big((S^3,\Gamma\cup L_1)_{L_2}\big) \ ,$$

where in the second expression we take L_1 to be part of the ribbon graph and colour it by Ω. We compute both sides from the definition:

$$\begin{aligned}\tau_{\mathcal{C}}\big((S^3,\Gamma)_{L_1\cup L_2}\big) &= \frac{\mathcal{D}}{\mathrm{Dim}(\mathcal{C})}\, \mathcal{D}^{-|L_1\cup L_2|}\, \delta^{-\sigma(L_1\cup L_2)}\, F_{\mathcal{C}}(\Gamma\cup L_1\cup L_2) \ ,\\ \tau_{\mathcal{C}}\big((S^3,\Gamma\cup L_1)_{L_2}\big) &= \frac{\mathcal{D}}{\mathrm{Dim}(\mathcal{C})}\, \mathcal{D}^{-|L_2|}\, \delta^{-\sigma(L_2)}\, F_{\mathcal{C}}(\Gamma\cup L_1\cup L_2) \ .\end{aligned}$$

Below we will need to be able to express the relation just in terms of L_1. As there is no simple relation between the signatures, we require

$$\delta = 1 \ .$$

Then

$$\tau_{\mathcal{C}}\big((S^3,\Gamma)_{L_1\cup L_2}\big) \ = \ \mathcal{D}^{-|L_1|}\, \tau_{\mathcal{C}}\big((S^3,\Gamma\cup L_1)_{L_2}\big) \ .$$

This motivates:

Definition 5.83. A twist-nondegenerate ribbon fusion category $\mathcal{C}$ together with a choice of $\mathcal{D}\in\Bbbk$ such that $\mathcal{D}^2 = \Delta_+(\mathcal{C})\Delta_-(\mathcal{C})$ is called *anomaly-free* if $\Delta_+ = \mathcal{D}$, or, equivalently, $\Delta_- = \mathcal{D}$.

In an anomaly-free theory we hence have $\delta = \Delta_+/\mathcal{D} = 1$, and since $\Delta_- = \mathcal{D}/\delta$ this is indeed equivalent to $\Delta_- = \mathcal{D}$.

Lemma 5.84. Let $\mathcal{C}$, $\mathcal{D}$ be anomaly-free. Let M be a (not necessarily connected) closed 3-manifold with embedded ribbon graph Γ, and let L be a surgery link in M. Then

$$\tau_{\mathcal{C}}\big((M,\Gamma)_L\big) = \mathcal{D}^{-|L|}\, \tau_{\mathcal{C}}\big((M,\Gamma\cup L)\big) \ .$$

Proof. We may assume that M is connected, else repeat the argument for each component. Consider $(M,\Gamma\cup L)$ and pick a surgery presentation $(M,\Gamma\cup L)\cong(S^3,\Gamma\cup L)_{L'}$. Then

$$\tau_{\mathcal{C}}\big((M,\Gamma\cup L)\big) = \tau_{\mathcal{C}}\big((S^3,\Gamma\cup L)_{L'}\big) \ .$$

Omitting L, we also have $(M,\Gamma) \cong (S^3,\Gamma)_{L'}$. By construction L is disjoint from Γ and L' and we can carry out surgery at L on both sides: $(M,\Gamma)_L \cong ((S^3,\Gamma)_{L'})_L = (S^3,\Gamma)_{L'\cup L}$. From this we get

$$\tau_{\mathcal{C}}\big((M,\Gamma)_L\big) = \tau_{\mathcal{C}}\big((S^3,\Gamma)_{L'\cup L}\big) \ .$$

□

Remark 5.85. The global dimension and Gauss sum are multiplicative in the sense that for two twist-nondegenerate ribbon fusion categories $\mathcal{B},\mathcal{C}$ we have (Why?) ?

$$\mathrm{Dim}(\mathcal{B}\boxtimes\mathcal{C}) = \mathrm{Dim}(\mathcal{B})\ \mathrm{Dim}(\mathcal{C}) \quad , \qquad \Delta_\pm(\mathcal{B}\boxtimes\mathcal{C}) = \Delta_\pm(\mathcal{B})\,\Delta_\pm(\mathcal{C}) \ .$$

Looking at Example 5.46 we see that for example $\mathsf{Is}^{\boxtimes 16}$ with $\mathcal{D}=2$ is anomaly free, as is $\mathsf{Fib}_{\varepsilon,\nu=+}\boxtimes\mathsf{Fib}_{\varepsilon,\nu=-}$ with positive square root $\mathcal{D}$, or $(\mathsf{Fib}_{\varepsilon,\nu})^{\boxtimes 10}$ with negative choice of square root.

For info: The condition $\delta=1$ of being anomaly-free can be dropped if one makes the bordism category more complicated: Objects Σ are equipped in addition with a Lagrangian subspace of $H_1(\Sigma,\mathbb{R})$, and bordisms with an integer. In the composition of two bordisms, the two integers add, but there is an additional contribution from the Lagrangian subspaces – see [Tu, Sec. IV.6] for details. From a different point of view, in surgery theory we are really considering how the boundary of a 4-manifold changes under handle attachments, and we should really be working with this 4d TFT [Wa, CGHP]. For $\delta=1$ this does not happen.

Step 3: Formal span of bordisms and canonical pairing

Fix an object $X\in\mathcal{S}$. We will define two functors

$$V_X, V'_X : \mathsf{Bord}_3(\mathcal{C}) \to \mathsf{Vec} \ ,$$

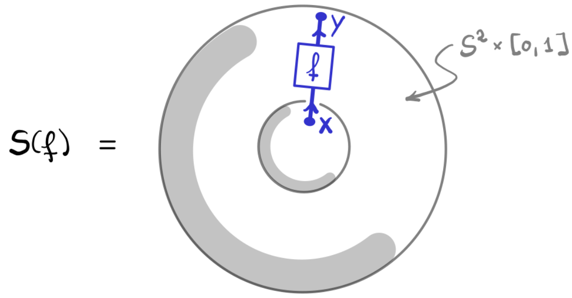
where V_X is covariant and V'_X is contravariant.

- Let

$$S(X)\in\mathsf{Bord}_3(\mathcal{C})$$

be the 2-sphere, identified with $\mathbb{R}^2\cup\{\infty\}$ and marked point $(X,+)$ at zero (and $*_{1,2}$ strata as above Definition 5.56). Analogously, for a morphisms

$f : X \to Y$ in $\mathcal{S}$, define $S(f) : S(X) \to S(Y)$ to be $S^2 \times [0,1]$ with a vertical ribbon with coupon f in the middle,



Note that this does not define a functor as $S(f \circ g) \neq S(f) \circ S(g)$ – the left hand side as one coupon $f \circ g$, the right hand side has two coupons.

■ Given $\Sigma \in \mathsf{Bord}_3(\mathcal{C})$, the vector spaces $V_X(\Sigma)$ and $V'_X(\Sigma)$ are the (very) infinite-dimensional formal spans over the full set of isomorphism classes of bordisms $S(X) \to \Sigma$ and $\Sigma \to S(X)$, respectively,

$$\begin{aligned} V_X(\Sigma) &= \mathrm{span}_{\Bbbk}\big\{ [W] \,\big|\, W : S(X) \to \Sigma \ , \ W \text{ connected} \big\} \ , \\ V'_X(\Sigma) &= \mathrm{span}_{\Bbbk}\big\{ [W'] \,\big|\, W' : \Sigma \to S(X) \ , \ W' \text{ connected} \big\} \ . \end{aligned}$$

Given a bordism $M : \Sigma \to \Sigma'$ (not necessarily connected), we define

$$\begin{aligned} V_X(M) : V_X(\Sigma) \to V_X(\Sigma') \quad &, \quad [W] \mapsto \big[S(X) \xrightarrow{W} \Sigma \xrightarrow{M} \Sigma'\big] \\ V'_X(M) : V'_X(\Sigma') \to V'_X(\Sigma) \ , \quad &, \quad [W'] \mapsto \big[\Sigma \xrightarrow{M} \Sigma' \xrightarrow{W'} S(X)\big] \ . \end{aligned}$$

We should really write $V_X([M])$ instead of $V_X(M)$, but there are already so many brackets. The above equations specify $V_X(M)$ on the defining basis of $V_X(\Sigma)$, and we extend it linearly.

V_X and V'_X are automatically into functors. (Why is it compatible with composition?) One feature of the universal construction of TFTs is that functoriality is for free, but we will have to work to get monoidality. ? (30.6.)

■ As hinted at by the notation, for each $\Sigma \in \mathsf{Bord}_3(\mathcal{C})$, there is a pairing

$$\langle -,- \rangle_X \ : \ V'_X(\Sigma) \times V_X(\Sigma) \to \Bbbk \ , \quad \langle [A'],[B] \rangle = \tau_{\mathcal{C}}\big(\mathrm{tr}_{S(X)}(A' \circ B)\big)$$

Here, $A' \circ B$ is a bordism $S(X) \to S(X)$ and $\mathrm{tr}_{S(X)}(A' \circ B)$ denotes the closed 3-manifold obtained by taking the trace (via (co)evaluation bordisms) in the symmetric monoidal category of bordisms. Equivalently, one can just identify the in- and outgoing boundary component $S(X)$.

Let $M : \Sigma \to \Sigma'$ be a bordism and $f : X \to Y$ a morphism in $\mathcal{S}$. By construction (cyclicity of the trace) the pairing has the following symmetry property, for $B : S(Y) \to \Sigma$ and $A' : \Sigma' \to S(X)$,

$$\big\langle\, [A']\,,\, V_X(M)[B \circ S(f)] \,\big\rangle_X \;=\; \big\langle\, V'_X(M)[S(f) \circ A']\,,\, [B] \,\big\rangle_Y \ .$$

(Check this.) ?

For info: In the original universal construction in [BHMV], one uses bordisms $\emptyset \to \Sigma$ to span the state space, not $S(X) \to \Sigma$ as we are doing here. However, in our setting this only works on the nose if $\mathcal{C}$ is modular as otherwise the candidate TFT functor will turn out not to be monoidal.

Step 4: Quotient by the kernel of the pairing

The pairing above is highly degenerate, and our next step is to divide by its kernel. We define

$$N_X(\Sigma) = \big\{ v \in V_X(\Sigma) \,\big|\, \langle w, v\rangle_X = 0 \text{ for all } w \in V'_X(\Sigma) \,\big\} \ ,$$

and

$$\mathbb{V}(X, \Sigma) = V_X(\Sigma)/N_X(\Sigma) \ .$$

For $W : S(X) \to \Sigma$ connected we write

$$[W] \in V_X(\Sigma) \quad \text{and} \quad [\![W]\!] = [W] + N_X(\Sigma) \in \mathbb{V}(X, \Sigma) \ .$$

Lemma 5.86. For $f : X \to Y$ and $M : \Sigma \to \Sigma'$ the map

$$\mathbb{V}(f, M) : \mathbb{V}(Y, \Sigma) \to \mathbb{V}(X, \Sigma') \quad , \quad [\![W]\!] \mapsto [\![M \circ W \circ S(f)]\!]$$

is well-defined and the unique map which makes the following diagram commute:

$$\begin{tikzcd} V_Y(\Sigma) \arrow[r, "\pi"] \arrow[d, "h"] & \mathbb{V}(Y, \Sigma) \arrow[d, dashed, "\exists!\, \mathbb{V}(f,M)"] \\ V_X(\Sigma') \arrow[r, "\pi"] & \mathbb{V}(X, \Sigma') \end{tikzcd}$$

where h is defined on the basis as

$$[W] \;\mapsto\; \big[\, \Sigma' \xleftarrow{M} \Sigma \xleftarrow{W} S(Y) \xleftarrow{S(f)} S(X) \,\big] \ .$$

Proof. Uniqueness is clear as the projections π are surjective. For well-definedness we need to check that $\pi \circ h$ vanishes on $N_Y(\Sigma)$. Let $v = \sum_\alpha \lambda_\alpha [W_\alpha] \in N_Y(\Sigma)$ (a linear combination of the basis elements $[W_\alpha]$. For a given $B' : \Sigma' \to S(X)$ we compute

$$\begin{aligned}
&\big\langle [B'], \pi(h(v)) \big\rangle_X = \sum_\alpha \lambda_\alpha \big\langle [B'], \pi(h([W_\alpha])) \big\rangle_X \\
&= \sum_\alpha \lambda_\alpha \big\langle [B'], [M \circ W_\alpha \circ S(f)] \big\rangle_X = \sum_\alpha \lambda_\alpha \big\langle [S(f) \circ B' \circ M], [W_\alpha] \big\rangle_Y \\
&\overset{(*)}{=} \big\langle [S(f) \circ B' \circ M], v \big\rangle_Y = 0 \ ,
\end{aligned}$$

where $(*)$ is the invariance property of the pairing at the end of step 3, and the last equality is just the assumption that $v \in N_Y(\Sigma)$. $\square$

As for V_X, we should really write $\mathbb{V}(f, [M])$ instead of $\mathbb{V}(f, M)$.

■ In the next lemma we will see that $\mathbb{V}$ is a functor $\mathcal{S} \times \mathsf{Bord}_3(\mathcal{C}) \to \mathsf{Vec}_\Bbbk$. If we fix one argument, for example $\mathbb{V}(X, -) : \mathsf{Bord}_3(\mathcal{C}) \to \mathsf{Vec}_\Bbbk$, by abuse of notation for a morphism $M : \Sigma \to \Sigma'$ we write $\mathbb{V}(X, M) : \mathbb{V}(X, \Sigma) \to \mathbb{V}(X, \Sigma')$ instead of $\mathbb{V}(1_X, M)$.

■ One can repeat the above considerations for the contravariant functor $V'_X : \mathsf{Bord}_3(\mathcal{C}) \to \mathsf{Vec}$:

$$\begin{aligned}
N'_X(\Sigma) &= \big\{ w \in V'_X(\Sigma) \,\big|\, \langle w, v \rangle_X = 0 \text{ for all } v \in V_X(\Sigma) \,\big\} \ , \\
\mathbb{V}'(X, \Sigma) &= V_X(\Sigma) / N_X(\Sigma) \ .
\end{aligned}$$

For $W' : \Sigma \to S(X)$ connected we write

$$[W'] \in V'_X(\Sigma) \quad \text{and} \quad [\![W']\!] = [W'] + N'_X(\Sigma) \in \mathbb{V}'(X, \Sigma) \ .$$

Remark 5.87. By construction, the pairing $\langle -, - \rangle_X$ descends to a pairing

$$\langle\!\langle -, - \rangle\!\rangle_X : \mathbb{V}'(X, \Sigma) \times \mathbb{V}(X, \Sigma) \to \Bbbk$$

uniquely defined by $\langle\!\langle [\![B']\!], [\![A]\!] \rangle\!\rangle_X = \langle [B'], [A] \rangle_X$. Again by construction, this pairing is non-degenerate in both arguments. (Details?) ?

Lemma 5.88. $\mathbb{V}, \mathbb{V}' : \mathcal{S} \times \mathsf{Bord}_3(\mathcal{C}) \to \mathsf{Vec}_\Bbbk$ are functors. $\mathbb{V}$ is contravariant in $\mathcal{S}$ and covariant in $\mathsf{Bord}_3(\mathcal{C})$, while $\mathbb{V}'$ is covariant in $\mathcal{S}$ and contravariant in $\mathsf{Bord}_3(\mathcal{C})$.

Proof. In both cases, functoriality in $\mathsf{Bord}_3(\mathcal{C})$ is clear as already V_X and V'_X are functorial.

We will show functoriality in $\mathcal{S}$ for $\mathbb{V}$, the argument for $\mathbb{V}'$ is similar. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ and Σ be given. Let $A : S(Z) \to \Sigma$ be arbitrary. We need to show $u = v$, where

$$\begin{aligned} u &= \mathbb{V}(g \circ f, \Sigma)([\![A]\!]) = [\![S(X) \xrightarrow{S(g\circ f)} S(Z) \xrightarrow{A} \Sigma]\!] \ , \\ v &= \mathbb{V}(f, \Sigma)\Big(\mathbb{V}(g, \Sigma)([\![A]\!])\Big) = \mathbb{V}(f, \Sigma)([\![A \circ S(g)]\!]) \\ &= [\![S(X) \xrightarrow{S(f)} S(Y) \xrightarrow{S(g)} S(Z) \xrightarrow{A} \Sigma]\!] \ , \end{aligned}$$

By non-degeneracy of the pairing (Remark 5.87) it is enough to check that for all $w = [\![B']\!]$ we have $\langle\!\langle w, u\rangle\!\rangle_X = \langle\!\langle w, v\rangle\!\rangle_X$. Explicitly,

$$\begin{aligned} \langle\!\langle w, u\rangle\!\rangle_X &= \tau_{\mathcal{C}}\Big(\mathrm{tr}_{S(X)}\big(S(X) \xrightarrow{S(g\circ f)} S(Z) \xrightarrow{A} \Sigma \xrightarrow{B'} S(X)\big)\Big) \ , \\ \langle\!\langle w, v\rangle\!\rangle_X &= \tau_{\mathcal{C}}\Big(\mathrm{tr}_{S(X)}\big(S(X) \xrightarrow{S(f)} S(Y) \xrightarrow{S(g)} S(Z) \xrightarrow{A} \Sigma \xrightarrow{B'} S(X)\big)\Big) \ . \end{aligned}$$

Finally, inside the Reshetikhin-Turaev functor we have $S(g \circ f) = S(g) \circ S(f)$. $\square$

Step 5: Finite-dimensionality

While the vector space $V_X(\Sigma)$ is uncountably infinite-dimensional, in step 5 we want to show:

Proposition 5.89. $\mathbb{V}(X, \Sigma)$ is finite-dimensional for all X, Σ.

The proof requires some preparation.

Lemma 5.90. Let $M : \Sigma \to \Sigma'$. Then

$$\mathbb{V}\big(X, (M, \Gamma)_L\big) \ = \ \mathcal{D}^{-|L|}\, \mathbb{V}\big(X, (M, \Gamma \cup L)\big) \ .$$

Proof. We check the condition in Remark 5.87. Let $A : S(X) \to \Sigma$ and $B' : \Sigma' \to S(X)$ be arbitrary. We have for $w = [\![B']\!]$,

$$\begin{aligned} \langle\!\langle w, \mathbb{V}\big(X, (M, \Gamma)_L\big)[\![A]\!]\rangle\!\rangle_X &= \tau_{\mathcal{C}}\big(\mathrm{tr}_{S(X)}(B' \circ (M, \Gamma)_L \circ A)\big) \ , \\ \mathcal{D}^{-|L|}\, \langle\!\langle w, \mathbb{V}\big(X, (M, \Gamma \cup L)\big)[\![A]\!]\rangle\!\rangle_X &= \mathcal{D}^{-|L|}\, \tau_{\mathcal{C}}\big(\mathrm{tr}_{S(X)}(B' \circ (M, \Gamma \cup L) \circ A)\big) \ . \end{aligned}$$

Note that B' and A contain their own ribbon graph, and so for example $B' \circ (M, \Gamma)_L \circ A = (B' \circ M \circ A, \Gamma_{B'} \cup \Gamma \cup \Gamma_A)_L$. Indeed, the surgery takes place just inside M and we may apply it before or after composition without changing the result.

The two expressions above are then equal by Lemma 5.84. $\square$

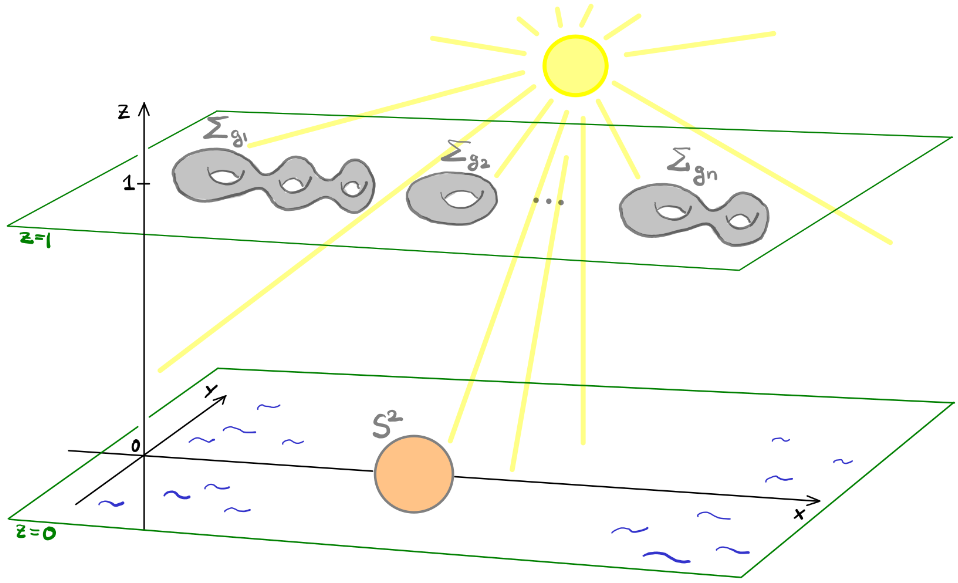


Figure 5.2: $B(\Sigma)$ looks like an island on a calm sea with genus-g clouds.

- By definition, $\mathbb{V}(X, \Sigma)$ is spanned by $[\![A]\!]$ with $A : S(X) \to \Sigma$, and

$$[\![A]\!] = \mathbb{V}(X, A)[\![1_{S(X)}]\!] \ .$$

Hence, the relation in Lemma 5.90 can be applied also to elements of $\mathrm{V}(X, \Sigma)$: For $(A, \Gamma) : S(X) \to \Sigma$,

$$[\![(A, \Gamma)_L]\!] = \mathcal{D}^{-|L|} \, [\![(A, \Gamma \cup L)]\!] \ .$$

This will be the key relation in the proof that $\mathbb{V}(X, \Sigma)$ is finite-dimensional.

- We need to import another result on surgery from topology:

Theorem 5.91 ([Ro])**.** Let $M, N : \Sigma_1 \to \Sigma_2$ be connected bordisms without embedded ribbon graphs. Then there exists a surgery link L in M such that $M_L \cong N$ as bordisms $\Sigma_1 \to \Sigma_2$.

We will use this to express all states in $\mathbb{V}(X, \Sigma)$ via S^3 with some pieces removed and added ribbon graphs. In more detail, let Σ_g be a genus-g surface, and H_g the corresponding handle body with $\partial H_g = \Sigma_g$. If Σ has no marked points, we can write

$$\Sigma \cong \Sigma_{g_1} \sqcup \cdots \sqcup \Sigma_{g_n}$$

We define

$$B(\Sigma) = S^3 \setminus (H_{g_1} \sqcup \cdots \sqcup H_{g_n} \sqcup B^3) \ ,$$

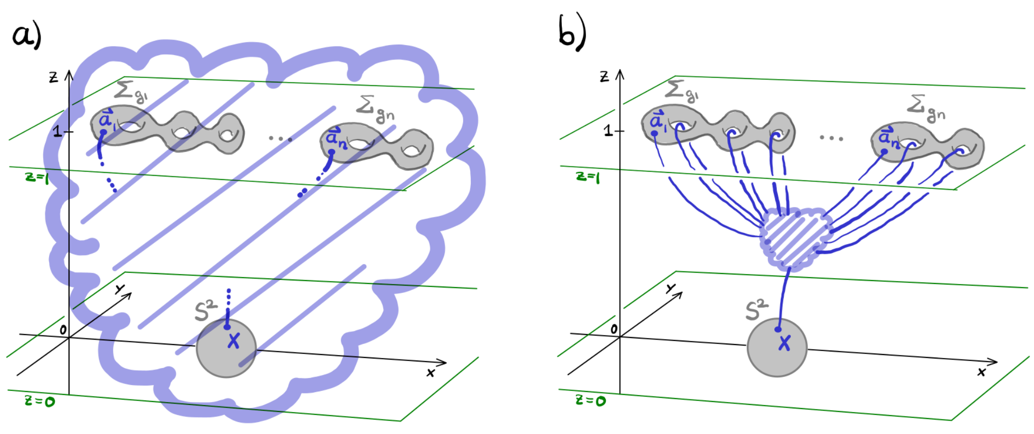


Figure 5.3: a) $B(X,\Sigma,\Gamma) : S(X) \to \Sigma$ with ribbon graph Γ indicated by the blue fog; b) deformation of the ribbon graph, with a single ribbon through each hole of the Σ_{g_i} – in case of several ribbons, these have been replaced by a single ribbon labelled by the corresponding tensor product.

where we think of the H_{g_i} as inserted around the plane $z = 1$, and B^3 inserted on the plane $z = 0$, see Figure 5.2.

At this point $B(\Sigma)$ is a bordism $S^2 \to \Sigma$. We still need to include the marked point X on S^2, marked points on Σ and a ribbon graph Γ. We write $\Sigma_{g,\vec{a}}$ for a genus-g surface with marked points $\vec{a} = ((a_1, \pm), \dots)$. In diagrams, we will for convenience collect this into one point labelled $\vec{a}$. For

$$\Sigma \cong \Sigma_{g_1,\vec{a}_1} \sqcup \cdots \sqcup \Sigma_{g_n,\vec{a}_n}$$

we define

$$B(X,\Sigma,\Gamma) = (B(\Sigma),\Gamma) : S(X) \to \Sigma \ ,$$

where Γ is a ribbon graph connected to the marked points X and $\vec{a}_1, \dots, \vec{a}_n$, see Figure 5.3 (a).

We have the following corollary to Theorem 5.91.

Corollary 5.92. Let $(M,\Gamma) : S(X) \to \Sigma$ be a connected bordism with embedded ribbon graph Γ. Then there is an isomorphism

$$(M,\Gamma) \cong B(X,\Sigma,\Gamma')_L$$

of bordisms $S(X) \to \Sigma$ for a suitable ribbon graph Γ' in $B(\Sigma)$.

Proof. As both M and $B(\Sigma)$ are connected bordisms $S(X) \to \Sigma$, from Theorem 5.91 we get a surgery link L such that $M \cong B(\Sigma)_L$. By isotoping the surgery link if necessary, we may assume that the ribbon graph Γ in M lies outside of a neighbourhood of the surgery link. We then take Γ' in $B(\Sigma)$ to be Γ, transported by the isomorphism. By construction, Γ' is disjoint from L and we have $(M, \Gamma) \cong B(X, \Sigma, \Gamma')_L$. $\square$

■ We need to bring the ribbon graph Γ in $B(X, \Sigma, \Gamma)$ into a standard form. To this end we need more notation:

- $G = \bigoplus_{a \in \mathcal{I}} a$ is the minimal generator $\mathcal{C}$, as already used in the definition of the surgery colour above Remark 5.42,
- $E = \bigoplus_{a \in \mathcal{I}} a^* \otimes a$, and
- $j : E \hookrightarrow G^* \otimes G$ is the diagonal embedding, that is, j maps the summand $a^* \otimes a \subset E$ identically to the summand $a^* \otimes a \subset G^* \otimes G$.

Recall from above Definition 4.23 the notation $T(\vec{a}) = a_1^{\varepsilon_1} \otimes \cdots \otimes a_n^{\varepsilon_n}$, for $\vec{a} = ((a_1, \varepsilon_1), \ldots, (a_n, \varepsilon_n))$. For a morphism

$$f : X \to (T(\vec{a}_1) \otimes E^{\otimes g_1}) \otimes \cdots \otimes (T(\vec{a}_n) \otimes E^{\otimes g_n})$$

we consider $B(X, \Sigma, \Gamma)$ for the special ribbon graph with a single coupon labelled f as shown in Figure 5.4. By abuse of notation, we will denote this bordism by

$$B(X, \Sigma; f) : S(X) \to \Sigma \ .$$

Lemma 5.93. For each Γ there is f such that

$$[\![B(X, \Sigma; \Gamma)]\!] = [\![B(X, \Sigma; f)]\!] \ \in \ \mathbb{V}(X, \Sigma) \ .$$

Proof. In Figure 5.3 (a), first deform the ribbon graph Γ to lie lear the plane $z = \frac{1}{2}$. In that process, part of the graph will get stuck at the holes of the boundary surfaces Σ_{g_i}, and we may assume that there is a single ribbon through each hole by using the tensor product to combine distinct ribbons. We arrive at the diagram shown in Figure 5.3 (b).

Next note that as G is a generator, each object $X \in \mathcal{C}$ can be written as a retract of $G^{\oplus m}$ for sufficiently large m, $e : G^{\oplus m} \leftrightarrow X : r$. Inserting such a retract on each ribbon looping through a hole of Σ_{g_i}, we may assume that all such ribbons are labelled G (resulting in a sum of ribbon graphs).

The sum of ribbon graph near the plane $z = \frac{1}{2}$ can now be collected into a single coupon f as in Figure 5.4. To see that f factors through E note that $b^* \otimes a \hookrightarrow G^* \otimes G$ gives zero for $b \neq a$ as the G-ribbon loops back onto itself. $\square$

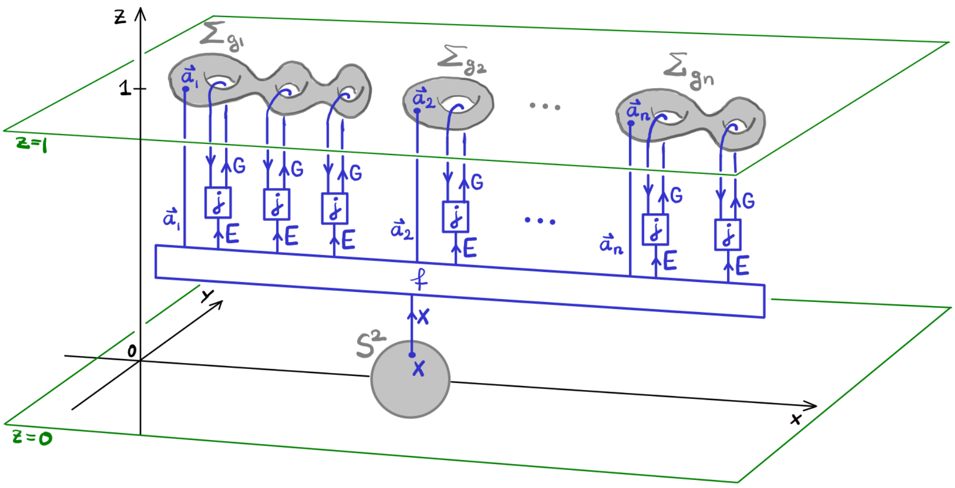


Figure 5.4: $B(X, \Sigma; f) : S(X) \to \Sigma$ has an embedded ribbon graph as shown, with a single coupon labelled f.

■ After this preparation, we can finally show that $\mathbb{V}(X, \Sigma)$ is finite-dimensional. In fact, we will show the following stronger result, which implies Proposition 5.89.

Proposition 5.94. For $X \in \mathcal{S}$ and $\Sigma = \Sigma_{g_1,\vec{a}_1} \sqcup \cdots \sqcup \Sigma_{g_n,\vec{a}_n}$, the map

$$\Phi_X : \mathcal{C}\Big(X\,,\,\bigotimes_{i=1}^{n}(T(\vec{a}_i) \otimes E^{\otimes g_i})\Big) \to \mathbb{V}(X, \Sigma) \quad , \quad f \mapsto [\![B(X, \Sigma, f)]\!]$$

is surjective and natural in X.

Proof. For surjectivity, let $(M, \Gamma) : S(X) \to \Sigma$ be given. By Corollary 5.92, we have $(M, \Gamma) \cong B(X, \Sigma, \Gamma')_L$ for suitable Γ' and surgery link L, and so in particular the first equality in

$$[\![(M, \Gamma)]\!] = [\![B(X, \Sigma, \Gamma')_L]\!] = \mathcal{D}^{-|L|}\,[\![B(X, \Sigma, \Gamma' \cup L)]\!] = [\![B(X, \Sigma; f)]\!] \ .$$

The second equality is the observation below Lemma 5.90, the third equality holds by Lemma 5.93 for a suitable f, where we also absorbed the prefactor $\mathcal{D}^{-|L|}$ into f. (1.7.26)

For naturality in X, abbreviate $Q = \bigotimes_{i=1}^{n}(T(\vec{a}_i) \otimes E^{\otimes g_i})$ and let $h : Y \to$

X. The naturality square is

$$\begin{array}{ccc}
\mathcal{C}(X,Q) & \xrightarrow{(\ )\circ h} & \mathcal{C}(Y,Q) \\
\downarrow{\scriptstyle \Phi_X} & & \downarrow{\scriptstyle \Phi_Y} \\
\mathbb{V}(X,\Sigma) & \xrightarrow{\mathbb{V}(h,\Sigma)} & \mathbb{V}(Y,\Sigma)
\end{array}$$

Starting with $f \in \mathcal{C}(X,Q)$, along the two paths we find

$$\begin{array}{ll}
\text{top:} & f \mapsto f\circ h \mapsto \Phi_Y(f\circ h) = [\![B(Y,\Sigma;f\circ h)]\!] \ , \\
\text{bottom:} & f \mapsto \Phi_X(f) \mapsto \mathbb{V}(h,\Sigma)(\Phi_X(f)) = [\![B(X,\Sigma;f)\circ S(h))]\!] \ .
\end{array}$$

These two are equal by a computation similar to the one used in the proof in Lemma 5.88 of functoriality of $\mathbb{V}$ in the first argument. (Details?) □ ?

■ Again we can repeat the above construction for $\mathbb{V}'$. One defines a bordism

$$B'(X,\Sigma;f) : \Sigma \to S(X)$$

as a in Figure 5.4, but reflected at the x-y-plane. Instead of the embeddings $j : E \to G^* \otimes G$ this then involves projections

$$p : G^* \otimes G \to E$$

which send $b^* \otimes a$ to zero for $b \neq a$ and are the identity on $a^* \otimes a$.

By the same line of arguments as above, one shows that for $X \in \mathcal{S}$ and $\Sigma = \Sigma_{g_1,\vec{a}_1} \sqcup \cdots \sqcup \Sigma_{g_n,\vec{a}_n}$, the map

$$\Phi'_X : \mathcal{C}\Big(\bigotimes_{i=1}^{n} (T(\vec{a}_i) \otimes E^{\otimes g_i})\,,\, X\Big) \to \mathbb{V}'(X,\Sigma) \quad , \quad f \mapsto [\![B'(X,\Sigma;f)]\!]$$

is surjective and natural in X.

■ Consider the genus-g surface $\Sigma_{g,\vec{a}}$ with marked points $\vec{a}$. We have the surjective maps

$$\begin{aligned}
\Phi_X &: \mathcal{C}(X, T(\vec{a}) \otimes E^{\otimes g}) \to \mathbb{V}(X,\Sigma_{g,\vec{a}}) \ , \\
\Phi'_X &: \mathcal{C}(T(\vec{a}) \otimes E^{\otimes g}, X) \to \mathbb{V}'(X,\Sigma_{g,\vec{a}}) \ ,
\end{aligned}$$

but we would like to find a Hom-space of $\mathcal{C}$ that is isomorphic to $\mathbb{V}(X,\Sigma)$, not just a surjection.

To this end, consider the induced pairing

$$t(-,-) : \mathcal{C}(T(\vec{a}) \otimes E^{\otimes g}, X) \times \mathcal{C}(X, T(\vec{a}) \otimes E^{\otimes g}) \to \Bbbk \ ,$$

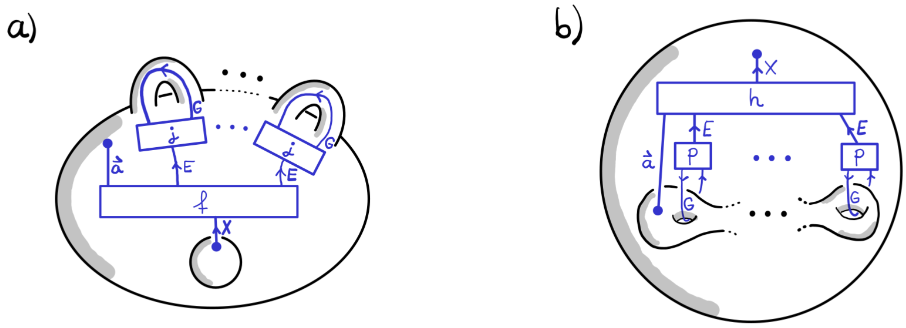


Figure 5.5: a) Alternative presentations of $B(X, \Sigma_{g,\vec{a}}; f) : S(X) \to \Sigma_{g,\vec{a}}$ as a handlebody with a 3-ball removed and with embedded ribbon graph as shown, and b) of $B'(X, \Sigma_{g,\vec{a}}; h) : \Sigma_{g,\vec{a}} \to S(X)$ as a 3-ball with embedded handlebody removed.

given by

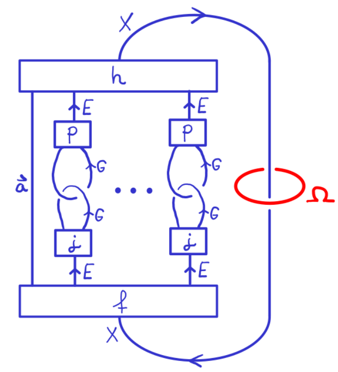
$$t(h, f) = \langle\!\langle \Phi'_X(h), \Phi_X(f) \rangle\!\rangle_X \ .$$

Writing out the definition, the pairing becomes

$$t(h, f) = \tau_{\mathcal{C}}\big(\mathrm{tr}_{S(X)}\big(S(X) \xrightarrow{\Phi_X(f)} \Sigma \xrightarrow{\Phi'_X(h)} S(X)\big)\big) \ .$$

We use the alternative presentations of $B(X, \Sigma_{g,\vec{a}}; f)$ and $B'(X, \Sigma_{g,\vec{a}}; h)$ shown in Figure 5.5. It is then easy to see that the underlying manifold of the bordism inside the trace is $S^2 \times [0, 1]$, so that after taking the trace one obtained $S^2 \times S^1$. The surgery presentation is an unknot in paper-plane framing (see Example 5.59 (2) and Exercise 5.79 (1)), which leads to the following expression:

$$t(h, f) = \frac{\mathcal{D}}{\mathrm{Dim}(\mathcal{C})}\, \mathcal{D}^{-1} F_{\mathcal{C}}\Bigg(\qquad \Bigg) \ .$$



We can simplify this a bit more. First, by construction, $X \in \mathcal{S} = Z_2(\mathcal{C})$ and so the Ω-loop can moved away from the X-ribbon and simplifies to a factor

$\mathrm{Dim}(\mathcal{C})$. Next, define the endomorphism

$\mathbb{S} =$  $: E \to E$.

Putting this together, we get

$$t(h,f) \;=\; \mathrm{tr}_X\big(h \circ (1_{T(\vec{a})} \otimes \mathbb{S}^{\otimes g}) \circ f\big) \ .$$

Interlude: Epi-mono factorisation, projective objects, tensor

■ *Epi-mono factorisation* of a morphism exists in any abelian category, but we did not introduce these, so consider for the moment $\mathcal{A} = R\text{-}\mathsf{mod}$, the category of left (say) R-modules. If X, Y are R-modules, and $f : X \to Y$ is an R-module morphism, let $\mathrm{im}(f) \subset Y$ be the image of f, which is an R-submodule of Y. Evidently, f factors as

$$f = \big[X \xrightarrow{\pi_f} \mathrm{im}(f) \xhookrightarrow{\iota_f} Y\big] \ ,$$

where π_f is surjective (and equal to f) and ι_f is injective (the embedding of the submodule). Epi-mono factorisations are unique up to unique isomorphism as the next exercise shows:

Exercise 5.95. For $f : X \to Y$ a morphism of R-modules, let E

$$f = \big[X \xrightarrow{\pi'} J \xhookrightarrow{\iota'} Y\big]$$

be another factorisation of f into an epi π (i.e. a surjective module morphism) and a mono ι (an injective module morphism). Show that there is a unique isomorphism $J \to \mathrm{im}(f)$ that makes the appropriate diagram commute. What does the epi-mono factorisation look like if f is an isomorphism?

(For info: Epi-mono factorisations in general abelian categories are described in [ML, Sec. VIII.1].)

■ *Projective objects:* Let again $\mathcal{A} = R\text{-}\mathsf{mod}$ (or any abelian category). An object $P \in \mathcal{A}$ is *projective* if we can "pull back morphisms along surjections", that is: For all $M, N \in \mathcal{A}$, $f : M \twoheadrightarrow N$ surjective, $h : P \to N$, there exists $\tilde{h} : P \to N$ such that $h = f \circ \tilde{h}$:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists \tilde{h}"'] \arrow[d, "h"] \\ M \arrow[r, two heads, "f"] & N \end{tikzcd}$$

Note that $\tilde{h}$ is in general not unique.

For example, let $R = k$ be a field. Then every $P \in k\text{-}\mathsf{mod} = \mathsf{Vec}_k$ is projective. (Why? *Hint:* Pick a basis of P, map it to M via h, and pick preimages in N. Use these to define the map $\tilde{h}$. Give an explicit example, where $\tilde{h}$ is not unique.) ?

As another example, let A be a finite-dimensional semisimple k-algebra. Then $M \in A\text{-}\mathsf{mod}^{\mathrm{fd}}$ is a direct sum $M = \bigoplus_{a \in \mathcal{I}} M_a \otimes_k a$ for some k-vector spaces M_a giving the multiplicity of the simple A-module a in M. All module homomorphism $f : M \to N$ are of the form $\bigoplus_{a \in \mathcal{I}} (M_a \xrightarrow{f_a} N_a)$, and we can apply the vector space argument componentwise to conclude that every $P \in A\text{-}\mathsf{mod}^{\mathrm{fd}}$ is projective.

(For info: R-mod is semisimple if and only if every R-module is projective.)

■ *Mono/epi and tensor:* Finally we need that the tensor product in rigid monoidal (and not necessarily linear or semisimple) categories preserves mono- and epimorphisms.

Lemma 5.96. Let $\mathcal{A}$ be a rigid monoidal category and let $a \in \mathcal{A}$. If $f : x \to y$ is mono (resp. epi), then also $1_a \otimes f$ and $f \otimes 1_a$ are mono (resp. epi).

Exercise 5.97. E

1. Show the above lemma. *Hint:* Consider the claim that $1_a \otimes f$ is mono. Given $p, q : b \to a \otimes x$ such that $(1_a \otimes f) \circ p = (1_a \otimes f) \circ q$ as morphisms $b \to a \otimes y$, you need to show that $p = q$. Use the duality maps to instead consider an equality of morphisms $a^* \otimes b \to y$ and then use that f is mono. Graphical calculus helps.

2. Let $f_i : X_i \to Y_i$, $i = 1, \dots, n$ be mono (resp. epi). Show that
$$f_1 \otimes \cdots \otimes f_n : X_1 \otimes \cdots \otimes X_n \to Y_1 \otimes \cdots \otimes Y_n$$
is mono (resp. epi).

This concludes the interlude. In our current setting $\mathcal{C}$ is in particular finitely semisimple, which means that it is equivalent to $A\text{-}\mathsf{mod}^{\mathrm{fd}}$ for A a finite-dimensional semisimple $\Bbbk$-algebra (Definition 5.19). Thus morphism in $\mathcal{C}$ admit epi-mono factorisations, and we apply this to $\mathbb{S}$:
$$\mathbb{S} = \left[E \xrightarrow{\pi_{\mathbb{S}}} D \xhookrightarrow{\iota_{\mathbb{S}}} E \right] ,$$
where $\pi_{\mathbb{S}}$ is epi and $\iota_{\mathbb{S}}$ is mono, and $D \in \mathcal{C}$ is an object which depends on $\mathbb{S}$.

Furthermore, all objects in $\mathcal{C}$ are projective, and so we get an $e_{\mathbb{S}} : D \to E$ which together with $\pi_{\mathbb{S}} : E \to D$ exhibits D as a retract of E:
$$\begin{tikzcd} & D \arrow[dl, dashed, "\exists e_{\mathbb{S}}"'] \arrow[d, "1_D"] \\ E \arrow[r, two heads, "\pi_{\mathbb{S}}"] & D \end{tikzcd}$$

We can now state the main result of this section:

Proposition 5.98. The map

$$\Psi_X : \mathcal{C}\big(X, T(\vec{a}) \otimes D^{\otimes g}\big) \to \mathbb{V}(X, \Sigma_{g,\vec{a}}) \quad , \quad u \mapsto \Phi_X\big((1_{T(\vec{a})} \otimes e_{\mathbb{S}}^{\otimes g}) \circ u\big) \ ,$$

is an isomorphism, natural in X.

Proof. Ψ_X *is surjective:* By surjectivity of Φ_X (Proposition 5.94 for $n = 1$), it is enough to show that for each $f : X \to T(\vec{a}) \otimes E^{\otimes g}$ there is $u : X \to T(\vec{a}) \otimes D^{\otimes g}$ such that $\Phi_X(f) = \Psi_X(u)$.

We start with an auxiliary claim. Write $p = e_{\mathbb{S}} \circ \pi_{\mathbb{S}} : E \to E$. We have:

$$\Phi_X(f) = \Phi_X(Q \circ f) \quad \text{for} \quad Q = (1_{T(\vec{a})} \otimes p^{\otimes g}) \ .$$

To see this, first note that by non-degeneracy of the pairing it is enough to check, for all h,

$$\langle\!\langle \Phi'_X(h), \Phi_X(f) \rangle\!\rangle_X = \langle\!\langle \Phi'_X(h), \Phi_X(Q \circ f) \rangle\!\rangle_X$$

or, equivalently by definition, $t(h, f) = t(h, Q \circ f)$. Using the explicit form of t before the interlude, we compute

$$\begin{aligned} t(h, Q \circ f) &= \mathrm{tr}_X\big(h \circ (1_{T(\vec{a})} \otimes \mathbb{S}^{\otimes g}) \circ (1_{T(\vec{a})} \otimes p^{\otimes g}) \circ f\big) \\ &\overset{(*)}{=} \mathrm{tr}_X\big(h \circ (1_{T(\vec{a})} \otimes \mathbb{S}^{\otimes g}) \circ f\big) = t(h, f) \ , \end{aligned}$$

where in $(*)$ we used that $\mathbb{S} \circ p = \mathbb{S}$. (Why does that hold?) This shows the claim. ?

To conclude the proof, we give a u such that $\Phi_X(f) = \Psi_X(u)$. Namely, we set

$$u \ = \ \big[\, X \xrightarrow{f} T(\vec{a}) \otimes E^{\otimes g} \xrightarrow{1_{T(\vec{a})} \otimes \pi_{\mathbb{S}}^{\otimes g}} T(\vec{a}) \otimes D^{\otimes g} \,\big] \ .$$

Then

$$\begin{aligned} \Psi_X(u) = \Phi_X\big((1_{T(\vec{a})} \otimes e_{\mathbb{S}}^{\otimes g}) \circ u\big) &= \Phi_X\big((1_{T(\vec{a})} \otimes e_{\mathbb{S}}^{\otimes g}) \circ (1_{T(\vec{a})} \otimes \pi_{\mathbb{S}}^{\otimes g}) \circ f\big) \\ &\overset{(*)}{=} \Phi_X(Q \circ f) = \Phi_X(f) \ , \end{aligned}$$

where $(*)$ is the definition of p and Q.

Ψ_X *is injective:* We need to show that $\langle\!\langle \Phi'_X(h), \Psi_X(u) \rangle\!\rangle_X = 0$ for all h implies $u = 0$. We have

$$\langle\!\langle \Phi'_X(h), \Psi_X(u) \rangle\!\rangle_X = \mathrm{tr}_X\big(h \circ (1_{T(\vec{a})} \otimes \mathbb{S}^{\otimes g}) \circ (1_{T(\vec{a})} \otimes e_{\mathbb{S}}^{\otimes g}) \circ u\big)$$

$$\overset{(*)}{=} \operatorname{tr}_X\big(h \circ (1_{T(\vec{a})} \otimes \iota_{\mathbb{S}}^{\otimes g}) \circ u\big) \ ,$$

where $(*)$ follows from $\mathbb{S} \circ e_{\mathbb{S}} = \iota_{\mathbb{S}} \circ \pi_{\mathbb{S}} \circ e_{\mathbb{S}} = \iota_{\mathbb{S}}$. Non-degeneracy of the trace pairing (Exercise 5.24 (6)) now implies that $(1_{T(\vec{a})} \otimes \iota_{\mathbb{S}}^{\otimes g}) \circ u = 0$, and since $\iota_{\mathbb{S}}$ is mono (using Exercise 5.97 (2)), we conclude $u = 0$.

Ψ_X is natural: By Proposition 5.94, Φ_X is natural in X. This means that, for each $h : Y \to X$ and $f : X \to T(\vec{a}) \otimes E^{\otimes g}$ we have

$$\Phi_Y(f \circ h) = \mathbb{V}(h, \Sigma_{g,\vec{a}})\big(\Phi_X(f)\big) \ .$$

Using this, we compute for Ψ:

$$\begin{aligned}\Psi_Y(u \circ h) &= \Phi_Y\big((1_{T(\vec{a})} \otimes e_{\mathbb{S}}^{\otimes g}) \circ u \circ h\big) \\ &= \mathbb{V}(h, \Sigma_{g,\vec{a}})\Big(\Phi_X\big((1_{T(\vec{a})} \otimes e_{\mathbb{S}}^{\otimes g}) \circ u\big)\Big) = \mathbb{V}(h, \Sigma_{g,\vec{a}})\big(\Psi_X(u)\big) \ .\end{aligned}$$

□

(7.7.26)

Exercise 5.99. Let $\mathcal{C}$ be a ribbon fusion category. Show that the following are equivalent: E

1. $\mathcal{C}$ is modular.
2. $\mathbb{S} : E \to E$ is invertible.

Hint: For 1⇒2 use that $\mathcal{I}_{\text{trans}} = \{\mathbb{1}\}$ and guess an inverse for $\mathbb{S}$. For 2⇒1 write $E = \bigoplus_{c \in \mathcal{I}} c^{\oplus n_c}$ and conclude that $\mathbb{S}$ is invertible iff each $n_c \times n_c$ matrix obtained from restricting $\mathbb{S}$ to $c^{\oplus n_c}$ is invertible. Then consider the case $c = \mathbb{1}$.

Exercise 5.100. Let $\mathcal{C}$ be an anomaly-free twist-nondegenerate ribbon fusion category (with square root $\mathcal{D}$). E

1. Assume that $\mathcal{C}$ is modular. Show that $D = E$ and hence that $\Phi_X : \mathcal{C}(X, T(\vec{a}) \otimes E^{\otimes g}) \to \mathbb{V}(X, \Sigma_{g,\vec{a}})$ is an isomorphism.
2. Assume that $\mathcal{C}$ is symmetric. Show that

$$\mathbb{S} = \big[E \xrightarrow{\pi_{\mathbb{S}}} \mathbb{1} \xrightarrow{\iota_{\mathbb{S}}} E\big] \ ,$$

with $\pi_{\mathbb{S}} = \sum_{a \in \mathcal{I}} \operatorname{ev}_a$ and a corresponding morphism for $\iota_{\mathbb{S}}$. Conclude that $\Psi_X : \mathcal{C}(X, T(\vec{a})) \to \mathbb{V}(X, \Sigma_{g,\vec{a}})$ is an isomorphism.

5.8 Representing objects and monoidality

Definition 5.101. Let $\mathcal{A}$ be a k-linear category. A covariant (resp. contravariant) functor $F : \mathcal{A} \to \mathsf{Vec}_k$ is *representable* if there is a natural isomorphism

$$\eta : \mathcal{A}(U, -) \Rightarrow F \qquad \big(\text{resp. } \eta : \mathcal{A}(-, U) \Rightarrow F\,\big)$$

for some object U called the *representing object*.

Example 5.102.
1. Let $\mathcal{C}$ be an anomaly-free and twist-nondegenerate ribbon fusion category. Suppose that $\mathcal{C}$ is in addition symmetric. Then $\mathbb{V}(-, \Sigma_{g,\vec{a}})$ is representable by Proposition 5.98 with representing object $T(\vec{a})$ (recall from Exercise 5.100 that $D = \mathbb{1}$ for $\mathcal{C}$ symmetric). (Why does that proposition not also show that the representing object is $T(\vec{a}) \otimes D^{\otimes g}$ if we drop the assumption that $\mathcal{C}$ is symmetric?) ?

2. Let k be a field and $A = k[X]$ the polynomial ring over k. Take $\mathcal{A} = A\text{-}\mathsf{mod}$. Let k_0 be the A-module with underlying vector space k on which X acts as zero. Consider the functor

$$F : A\text{-}\mathsf{mod} \to \mathsf{Vec}_k \quad , \quad M \mapsto M \otimes_A k_0 \ .$$

Then F is not representable as F is right exact but not exact and the Hom-functor is left exact. For example, $f : A \to A$ with $f(a) = Xa$ is an injective A-module morphism of A seen as a left module over itself. But

$$A \otimes_A k_0 \xrightarrow{f \otimes_A \mathrm{id}_k} A \otimes_A k_0$$

is not injective and is in fact the zero map. (Why? *Hint:* $A \otimes_A k_0 \cong k_0$ via the map $a \otimes_A \lambda \mapsto a.\lambda$. What does the above map look like after applying this isomorphism to both sides?) ? On the other hand,

$$\mathrm{Hom}_A(U, A) \to \mathrm{Hom}_A(U, A) \quad , \quad h \mapsto f \circ h \ ,$$

is injective as f is injective. Altogether, F maps the morphism $f : A \to A$ to zero, but $\mathrm{Hom}_A(U, -)$ only maps f to zero if U is zero. (Why is 0 not the representing object of F?) ?

Remark 5.103.
1. More fundamentally, for a general category C, a functor $F : C \to \mathsf{Set}$ is representable if it is equivalent to $C(U, -)$ or $C(-, U)$. But we will work k-linearly, so we introduced that version of representability.

2. A representing object is unique up to unique isomorphism. For example, if F is covariant and also U' is a representing object via $\eta' : \mathcal{A}(U',-) \xRightarrow{\sim} F$, then there exists a unique morphism $f : U' \to U$ such that

$$\begin{tikzcd} \mathcal{A}(U,X) \arrow[dd, dashed, "(-)\circ f"'] \arrow[dr, "\eta_X"] & \\ & F(X) \\ \mathcal{A}(U',X) \arrow[ur, "\eta'_X"'] & \end{tikzcd}$$

commutes for all X. This f is an isomorphism.

3. Let $F, F' : \mathcal{A} \to \mathsf{Vec}_k$ be contravariant and representable with representing objects U, U'. Let $\phi : F \Rightarrow F'$ be a natural transformation. Then there is a unique morphism $Y(\phi) : U \to U'$ such that the following diagram commutes for all X:

$$\begin{tikzcd} \mathcal{A}(X,U) \arrow[r, "\eta_X"] \arrow[d, "Y(\phi)\circ(-)"'] & F(X) \arrow[d, "\phi_X"] \\ \mathcal{A}(X,U') \arrow[r, "\eta'_X"] & F'(X) \end{tikzcd}$$

This defines a k-linear isomorphism

$$Y \equiv Y_{F',F} : \mathrm{Nat}(F,F') \to \mathcal{A}(U,U')$$

which is compatible with composition: If $\psi : F' \Rightarrow F''$, then $Y(\psi \circ \phi) = Y(\psi) \circ Y(\phi)$ (omitting the indices of Y).

Exercise 5.104. Show parts 2 and 3 of the above remark. (TODO: add hint about Yoneda)

■ Let now $\Bbbk$ be algebraically closed and $\mathcal{A}$ be $\Bbbk$-linear and finitely semisimple. Then the map

$$\phi : \bigoplus_{a \in \mathcal{I}} \mathcal{A}(X,a) \otimes_{\Bbbk} \mathcal{A}(a,U) \to \mathcal{A}(X,U) \quad , \quad \bigoplus_a p_a \otimes_{\Bbbk} q_a \mapsto \sum_a q_a \circ p_a$$

is an isomorphism. (Why? *Hint:* Try $f \mapsto \sum_{a,\alpha} (r_{a,\alpha} \circ f) \otimes_{\Bbbk} e_{a,\alpha}$ for the inverse, where $e_{a,\alpha}$, $r_{a,\alpha}$ describe the direct sum $U = \bigoplus_a a^{\oplus n_a}$.) ?

For info: For a not algebraically closed field k we do not necessarily have $\mathcal{A}(a,a) = k\, 1_a$ but only that $\mathcal{A}(a,a)$ is a finite-dimensional division algebra over k. In this case we have to take $\otimes_{\mathcal{A}(a,a)}$ instead of $\otimes_{\Bbbk}$. The next lemma also holds over general fields, with an appropriately adapted proof taking the above modification into account.

Lemma 5.105. Let $\mathcal{A}$ be $\Bbbk$-linear and finitely semisimple. Then every $\Bbbk$-linear functor $\mathcal{A} \to \mathsf{Vec}_{\Bbbk}^{\mathrm{fd}}$ is representable.

Proof. Consider the case that $F : \mathcal{A} \to \mathsf{Vec}_{\Bbbk}^{\mathrm{fd}}$ is contravariant, the covariant case is analogous. For the representing object we make the ansatz

$$U = \bigoplus_{a \in \mathcal{I}} a \otimes_{\Bbbk} F(a) \ ,$$

where $a \in \mathcal{A}$ is simple and $F(a) \in \mathsf{Vec}_{\Bbbk}^{\mathrm{fd}}$. We can think of $F(a)$ as being the multiplicity space of a in U, or we can recall that $\mathcal{A}$ is given by modules over a semisimple $\Bbbk$-algebra, and $\otimes_{\Bbbk}$ is the tensor product of the underlying vector spaces. Our ansatz for the natural isomorphism

$$\eta_X : \mathcal{A}(X, U) \xrightarrow{\sim} F(X)$$

is as follows.

Recall the isomorphism ϕ defined before the lemma. Each $q \in \mathcal{A}(a, U)$ is (uniquely) of the form $q = 1_a \otimes_{\Bbbk} v_q$ for $v_q \in F(a)$. (Why? *Hint:* q has to map to the summand $a \otimes_{\Bbbk} F(a)$. For each linear form $\varphi : F(a) \to \Bbbk$, $(\mathrm{id}_a \otimes_{\Bbbk} \varphi) \circ q$ is proportional to id_a. Write id_a as a sum basis – dual basis.) For $p : X \to a$ and $q : a \to U$, $q = 1_a \otimes v_q$, we set ?

$$\eta_X(\phi(p \otimes_{\Bbbk} q)) = \big(F(p)\big)(v_q) \ .$$

Indeed, $F(p)$ is a linear map $F(a) \to F(X)$ as F is contravariant, and $v_q \in F(a)$.

We need to show that η_X is a natural isomorphism. Naturality will be an exercise. Given that η_X is natural, it is enough to check that it is an isomorphism for $X = b$ simple. (Why? *Hint:* Write X as a direct sum of simples with embedding and restriction maps $e_{b,\alpha}$, $r_{b,\alpha}$ and use naturality to write η_X in terms of η_b.) ?

For $p = 1_b$, $q = 1_b \otimes_{\Bbbk} v_q \in \mathcal{A}(b, U)$ with $v_q \in F(b)$ we have

$$\eta_b(q) = \eta_b(\phi(1_b \otimes_{\Bbbk} q)) = \big(F(1_b)\big)(v_q) = v_q \ .$$

As $v_q \mapsto q = 1_b \otimes_{\Bbbk} v_q$ is a bijection $F(b) \to \mathcal{A}(b, U)$, so is η_b. □

Exercise 5.106. Show that η_X in the above proof is natural in X.

■ Let us return to the setting in which we want to construct the TFT:

- $\Bbbk$: algebraically closed field,

- $\mathcal{C}$: anomaly-free twist-nondegenerate ribbon fusion category with square root $\mathcal{D}$,
- $\mathcal{S} = Z_2(\mathcal{C})$, the symmetric centre of $\mathcal{C}$,
- $\mathbb{V} : \mathcal{S} \times \mathsf{Bord}_3(\mathcal{C}) \to \mathsf{Vec}_{\Bbbk}^{\mathrm{fd}}$ is contravariant in the first argument and covariant in the second.

Let $\Sigma \in \mathsf{Bord}_3(\mathcal{C})$ be given. Then $\mathbb{V}(-,\Sigma) : \mathcal{S} \to \mathsf{Vec}_{\Bbbk}^{\mathrm{fd}}$ is a $\Bbbk$-linear contravariant functor. By Lemma 5.105, the functor is representable. Let us call the representing object $\tau_{\mathcal{C}}(\Sigma) \in \mathcal{S}$ with natural-in-X isomorphism

$$\eta(\Sigma)_X : \mathcal{S}\big(X, \tau_{\mathcal{C}}(\Sigma)\big) \xrightarrow{\sim} \mathbb{V}(X,\Sigma) \ .$$

Proposition 5.107. For $M : \Sigma \to \Sigma'$ in $\mathsf{Bord}_3(\mathcal{C})$ there exists a unique $\tau_{\mathcal{C}}(M) : \tau_{\mathcal{C}}(\Sigma) \to \tau_{\mathcal{C}}(\Sigma')$ such that

$$\begin{tikzcd} \mathcal{S}\big(X, \tau_{\mathcal{C}}(\Sigma)\big) \arrow[r, "\eta(\Sigma)_X"] \arrow[d, "\tau_{\mathcal{C}}(M)\circ(-)"'] & \mathbb{V}(X,\Sigma) \arrow[d, "\mathbb{V}(X,M)"] \\ \mathcal{S}\big(X, \tau_{\mathcal{C}}(\Sigma')\big) \arrow[r, "\eta(\Sigma')_X"] & \mathbb{V}(X,\Sigma') \end{tikzcd}$$

commutes for all $X \in \mathcal{S}$. This defines a functor

$$\tau_{\mathcal{C}} : \mathsf{Bord}_3(\mathcal{C}) \to Z_2(\mathcal{C}) \ .$$

Proof. Note that $\mathbb{V}(-,M)$ is a natural transformation $\mathbb{V}(-,\Sigma) \to \mathbb{V}(-,\Sigma')$. Hence, existence and uniqueness of $\tau_{\mathcal{C}}(M)$ follows from Remark 5.103 (3), as does functoriality. $\square$

Remark 5.108. 1. Recall the isomorphism Ψ_X from Proposition 5.98 and the computation of D in Exercise 5.100.

- $\mathcal{C}$ *symmetric:* $D = \mathbb{1}$ and we already computed the representing object in Example 5.102 (1) to be
$$\tau_{\mathcal{C}}(\Sigma_{g,\vec{a}}) = T(\vec{a}) \ .$$
- $\mathcal{C}$ *modular:* $D = E$ and $\mathcal{S} = \mathsf{Vec}_{\Bbbk}^{\mathrm{fd}}$. The representing object is
$$\tau_{\mathcal{C}}(\Sigma_{g,\vec{a}}) = \mathcal{C}(\mathbb{1}, T(\vec{a}) \otimes E^{\otimes g}) \ .$$

  Indeed, for all $X \in \mathsf{Vec}_{\Bbbk}^{\mathrm{fd}}$, naturally in X,
$$\mathrm{Hom}_{\Bbbk}(X, \tau_{\mathcal{C}}(\Sigma_{g,\vec{a}})) \cong \mathrm{Hom}_{\Bbbk}(X,\Bbbk) \otimes_{\Bbbk} \mathrm{Hom}_{\Bbbk}(\Bbbk, \tau_{\mathcal{C}}(\Sigma_{g,\vec{a}}))$$

$$\cong \mathrm{Hom}_{\Bbbk}(X, \Bbbk) \otimes_{\Bbbk} \mathcal{C}(\mathbb{1}, T(\vec{a}) \otimes E^{\otimes g})$$
$$\cong \mathcal{C}(X, T(\vec{a}) \otimes E^{\otimes g}) \cong \mathbb{V}(X, \Sigma_{g,\vec{a}}) \ ,$$

where the first isomorphism is the one before Lemma 5.105 and the last isomorphism is Ψ_X.

2. For all $\mathcal{C}$ we have
$$\tau_{\mathcal{C}}(S^2) = \mathbb{1} \in \mathcal{S} \ .$$
Indeed, $S^2 = \Sigma_{g,\vec{a}}$ with $g = 0$ and $\vec{a} = ()$, and so $T(\vec{a}) \otimes D^{\otimes g} = \mathbb{1}$. Then Ψ_X is an isomorphism $\mathcal{C}(X, \mathbb{1}) \to \mathbb{V}(X, S^2)$ and as $\mathbb{1} \in \mathcal{S} \subset \mathcal{C}$ we have $\mathcal{C}(X, \mathbb{1}) = \mathcal{S}(X, \mathbb{1})$. Altogether, for all $X \in \mathcal{S}$ we have the isomorphism, natural in X,
$$S(X, \mathbb{1}) = \mathcal{C}(X, \mathbb{1}) \xrightarrow{\Psi_X} \mathbb{V}(X, S^2) \ .$$

3. Recall from Step 1 in Section 5.7 that for a TFT we must have
$$\tau_{\mathcal{C}}(\Sigma \times S^1) = \dim \tau_{\mathcal{C}}(\Sigma) \ ,$$
where dim is the categorical dimension in $\mathcal{S}$. So far we have only shown that $\tau_{\mathcal{C}}$ is functorial, and not that it is symmetric monoidal. But we can still check the above identity in the cases where we computed the left hand side in Example 5.82.

   - $\mathcal{C}$ *symmetric:* We have $\tau_{\mathcal{C}}(\Sigma_{g,\vec{a}} \times S^1) = \dim T(\vec{a})$.
     This matches the result from the first point that $\tau_{\mathcal{C}}(\Sigma_{g,\vec{a}}) = T(\vec{a})$.
   - $\mathcal{C}$ *modular:* We have $\tau_{\mathcal{C}}(S^2 \times S^1) = 1$ and $\tau_{\mathcal{C}}(T^2 \times S^1) = |\mathcal{I}|$.
     For S^2 we saw in the previous point that $\tau_{\mathcal{C}}(S^2) = \mathbb{1}$, whose dimension is 1. For $T^2 = \Sigma_{1,()}$ we have
     $$\tau_{\mathcal{C}}(T^2) = \mathcal{C}(\mathbb{1}, E) = \mathcal{C}(\mathbb{1}, \bigoplus_{a \in \mathcal{I}} a^* \otimes a) \cong \bigoplus_{a \in \mathcal{I}} \underbrace{\mathcal{C}(\mathbb{1}, a^* \otimes a)}_{\cong \mathcal{C}(a,a) \cong \Bbbk} \ .$$
     And so indeed $\dim_{\Bbbk}(\tau_{\mathcal{C}}(T^2)) = |\mathcal{I}|$.

Monoidality of $\tau_{\mathcal{C}}$

We need to find natural isomorphisms
$$\tau^2_{\Sigma_1,\Sigma_2} : \tau_{\mathcal{C}}(\Sigma_1) \otimes_{\mathcal{S}} \tau_{\mathcal{C}}(\Sigma_2) \xrightarrow{\sim} \tau_{\mathcal{C}}(\Sigma_1 \sqcup \Sigma_2)$$

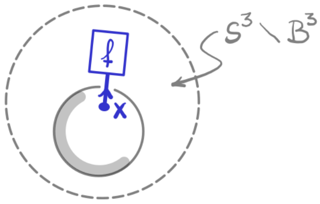
and an isomorphism

$$\tau^0 : \mathbb{1} \xrightarrow{\sim} \tau_{\mathcal{C}}(\emptyset) \ ,$$

so that the coherence diagrams in Definition 2.23 are satisfied. In the rest of this section, we will find these isomorphisms explicitly, but we will not verify the coherence diagrams.

The isomorphism τ^0 is easy to give. The map

$$\Phi_X : \underbrace{\mathcal{C}(X, \mathbb{1})}_{=\mathcal{S}(X,\mathbb{1}) \text{ as } X,\mathbb{1}\in\mathcal{S}} \to \mathbb{V}(X, \emptyset) \ , \quad f \mapsto [\![S(f)]\!] \quad , \quad S(f) =$$



is an isomorphism. (Why? *Hint:* Show first that $\langle\!\langle \Phi'_X(g), \Phi_X(f) \rangle\!\rangle_X = \mathrm{tr}_X(g \circ f)$.) ? Hence the representing object is $\tau_{\mathcal{C}}(\emptyset) = \mathbb{1}$ (with $\eta(\emptyset)_X = \Phi_X$). For τ^0 we can simply choose $\tau^0 = 1_{\mathbb{1}}$.

■ Before going into the technical details for τ^2, let us see what we need to do. We would like to use $\mathbb{V}$ and properties of representing objects to define τ^2. We know that $\tau_{\mathcal{C}}(\Sigma_1 \sqcup \Sigma_2)$ is the representing object of $\mathbb{V}(-, \Sigma_1 \sqcup \Sigma_2)$, but what is $\tau_{\mathcal{C}}(\Sigma_1) \otimes_{\mathcal{S}} \tau_{\mathcal{C}}(\Sigma_2)$ the representing object of? We know

$$\begin{aligned} \mathbb{V}(X^{(1)}, \Sigma_1) \otimes_{\Bbbk} \mathbb{V}(X^{(2)}, \Sigma_2) &\cong \mathcal{S}(X^{(1)}, \tau_{\mathcal{C}}(\Sigma_1)) \otimes_{\Bbbk} \mathcal{S}(X^{(2)}, \tau_{\mathcal{C}}(\Sigma_2)) \\ &\cong \mathcal{S}^{\boxtimes 2}(X^{(1)} \boxtimes X^{(2)}, \tau_{\mathcal{C}}(\Sigma_1) \boxtimes \tau_{\mathcal{C}}(\Sigma_2)) \end{aligned}$$

where $\mathcal{S}^{\boxtimes 2} = \mathcal{S} \boxtimes \mathcal{S}$ as introduced on page 157. The idea is now this: suppose the tensor product functor $T : \mathcal{S} \boxtimes \mathcal{S} \to \mathcal{S}$ had a left adjoint $L : \mathcal{S} \to \mathcal{S} \boxtimes \mathcal{S}$,

$$\mathcal{S}(A, T(Q)) \cong \mathcal{S}^{\boxtimes 2}(L(A), Q) \ .$$

Write $L(X) = \bigoplus_\alpha X_\alpha^{(1)} \boxtimes X_\alpha^{(2)}$. Then

$$\begin{aligned} \mathcal{S}(X, \underbrace{\tau_{\mathcal{C}}(\Sigma_1) \otimes_{\mathcal{S}} \tau_{\mathcal{C}}(\Sigma_2)}_{=T(\tau_{\mathcal{C}}(\Sigma_1) \boxtimes_{\mathcal{S}} \tau_{\mathcal{C}}(\Sigma_2))}) &\cong \mathcal{S}^{\boxtimes 2}(L(X), \tau_{\mathcal{C}}(\Sigma_1) \boxtimes \tau_{\mathcal{C}}(\Sigma_2)) \\ &\cong \bigoplus_\alpha \mathbb{V}(X_\alpha^{(1)}, \Sigma_1) \otimes_{\Bbbk} \mathbb{V}(X_\alpha^{(2)}, \Sigma_2) \end{aligned}$$

Thus our plan is to show that T has a left adjoint and that the above expression is naturally isomorphic to $\mathbb{V}(X, \Sigma_1 \sqcup \Sigma_2)$.

Interlude: Adjoint of the tensor product

Let $\mathcal{A}$ be a fusion category, and let $T : \mathcal{A}^{\boxtimes 2} \to \mathcal{A}$,

$$T(\textstyle\bigoplus_\alpha A_\alpha \boxtimes B_\alpha) = \bigoplus_\alpha A_\alpha \otimes B_\alpha \ .$$

Proposition 5.109. *T has a left and a right adjoint $L, R : \mathcal{A} \to \mathcal{A}^{\boxtimes 2}$, with natural isomorphisms, for all $X \in \mathcal{A}$, $P \in \mathcal{A}^{\boxtimes 2}$,*

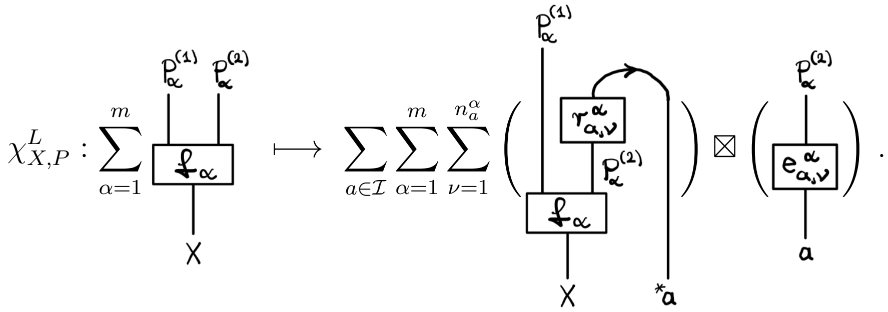
$$\chi^L_{X,P} : \mathcal{A}(X, T(P)) \xrightarrow{\sim} \mathcal{A}^{\boxtimes 2}(L(X), P) \ ,$$
$$\chi^R_{X,P} : \mathcal{A}(T(P), X) \xrightarrow{\sim} \mathcal{A}^{\boxtimes 2}(P, R(X)) \ .$$

We will describe L, R, and $\chi^{L/R}$ explicitly and then sketch the proof of the proposition.

- The functors are

$$L(X) = \bigoplus_{a \in \mathcal{I}} (X \otimes {}^*a) \boxtimes a \quad , \quad R(X) = \bigoplus_{a \in \mathcal{I}} (X \otimes a^*) \boxtimes a \ .$$

To describe the natural isomorphisms, let $P = \bigoplus_{\alpha=1}^m P_\alpha^{(1)} \boxtimes P_\alpha^{(2)}$ be given. Consider the direct sum decomposition $P_\alpha^{(2)} = \bigoplus_{a \in \mathcal{I}} a^{\oplus n_a^\alpha}$, realised by $r^\alpha_{a,\nu} : P_\alpha^{(2)} \to a$, $e^\alpha_{a,\nu} : a \to P_\alpha^{(2)}$, $\nu = 1, \dots, n_a^\alpha$. Then

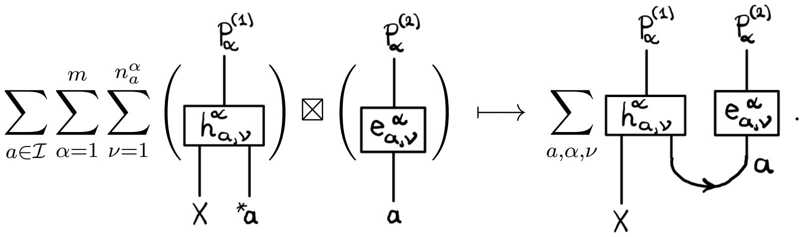


(The formula for $\chi^R_{X,P}$ is similar. Write it out.) ?

Proof of Proposition 5.109 (sketch). We need to check that $\chi^{L/R}_{X,P}$ is 1) independent of the choice of $e^\alpha_{a,\nu}$, $r^\alpha_{a,\nu}$, 2) an isomorphism for all X, P, and 3) natural in X and P.

We will not do any of this and just give the explicit inverse of $\chi^L_{X,P}$,

$$(\chi^L_{X,P})^{-1} \ : \ \mathcal{A}^{\boxtimes 2}(L(X), P) \xrightarrow{\sim} \mathcal{A}(X, T(P)) \ ,$$

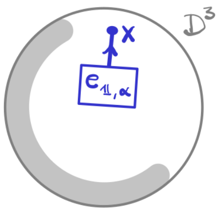
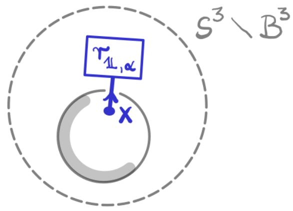


Since $e^{\alpha}_{a,\nu}$ forms a basis of $\mathcal{A}(a, P^{(2)}_\alpha)$, an element of $\mathcal{A}^{\boxtimes 2}(L(X), P)$ can always be expanded uniquely as on the left hand side. $\square$

Exercise 5.110. TODO 1) move f to other leg 2) indep of e,r and naturality 3) define $L'(X) = {}^*a \boxtimes (a \otimes X)$

This ends the short interlude on the adjoints of the tensor product functor.

■ For $X \in \mathcal{C}$ (not just $\mathcal{S}$) let $X = \bigoplus_{a\in\mathcal{I}} a^{\oplus n_a}$, realised by $r_{a,\alpha} : X \to a$, $e_{a,\alpha} : a \to X$. For $\alpha = 1, \dots, n_{\mathbb{1}}$, let $D(\alpha) : \emptyset \to S(X)$ and $D'(\alpha) : S(X) \to \emptyset$ be given by

$D(\alpha) =$  , $D'(\alpha) =$  .

Both are 3-discs, but drawing them this way makes it easier to compose them.

If the 2-sphere $S(X)$ can be embedded in a closed 3-manifold so that cutting along $S(X)$ decomposes it into two pieces, we can use $D(\alpha)$ and $D'(\alpha)$ to write the invariant of the full manifold as a sum of products:

Lemma 5.111. For $M : \emptyset \to S(X)$ and $N : S(X) \to \emptyset$ we have

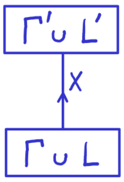
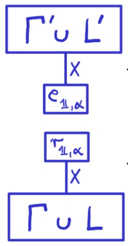
$$\tau_{\mathcal{C}}(N \circ M) = \frac{\operatorname{Dim} \mathcal{C}}{\mathcal{D}} \sum_{\alpha=1}^{n_{\mathbb{1}}} \tau_{\mathcal{C}}\big(N \circ D(\alpha)\big)\, \tau_{\mathcal{C}}\big(D'(\alpha) \circ M\big) \ .$$

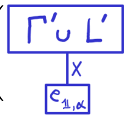
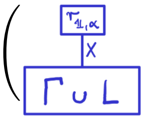
Proof. We may assume that M and N are connected (the components not connected to the boundary of one of M or N give the same exact factors on both sides). Pick surgery presentations

$$N \cong (S^3 \setminus B^3, \Gamma')_{L'} \quad \text{and} \quad M \cong (S^3 \setminus B^3, \Gamma)_L \ .$$

Then

$$\tau_{\mathcal{C}}(N \circ M) \overset{(1)}{=} \mathcal{D}^{-|L|-|L'|} \tau_{\mathcal{C}}(S^3, \Gamma \cup L \cup \Gamma' \cup L')$$

$\overset{(2)}{=} \dfrac{\mathcal{D}^{-|L|-|L'|+1}}{\operatorname{Dim}\mathcal{C}} F_{\mathcal{C}}\Bigg($  $\Bigg) \overset{(3)}{=} \dfrac{\mathcal{D}^{-|L|-|L'|+1}}{\operatorname{Dim}\mathcal{C}} \displaystyle\sum_{\alpha=1}^{n_{\mathbb{1}}} F_{\mathcal{C}}\Bigg($  $\Bigg)$

$\overset{(4)}{=} \dfrac{\operatorname{Dim}\mathcal{C}}{\mathcal{D}} \displaystyle\sum_{\alpha=1}^{n_{\mathbb{1}}} \tau_{\mathcal{C}}\Bigg($  $\Bigg) \tau_{\mathcal{C}}\Bigg($  $\Bigg)$

$$\overset{(5)}{=} \frac{\operatorname{Dim}\mathcal{C}}{\mathcal{D}} \sum_{\alpha=1}^{n_{\mathbb{1}}} \tau_{\mathcal{C}}\big(N \circ D(\alpha)\big)\, \tau_{\mathcal{C}}\big(D'(\alpha) \circ M\big) \ .$$

In step (1) we replace the closed 3-manifold $N \circ M$ by its surgery presentation and apply Lemma 5.84. Step (2) is Definition 5.81 of $\tau_{\mathcal{C}}$. (In step (3), why is there not also a sum over $a \in \mathcal{I}$, that is, why is it enough to take the contribution from $a = \mathbb{1}$?) In step (4) we apply $F_{\mathcal{C}}$ to the two disjoint component of the ribbon graph separately and rewrite each $F_{\mathcal{C}}$ as $\tau_{\mathcal{C}}$. Finally, in step (5) the surgery presentation is undone. □ (?)

(In general, the identity in the lemma is not true for $\Sigma_1 \xrightarrow{M} S(X) \xrightarrow{N} \Sigma_2$, but it does hold when $\mathcal{C}$ is a modular fusion category.)

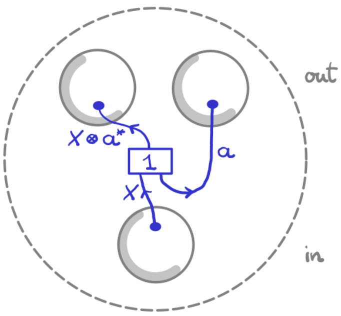
Corollary 5.112. For $a, b \in \mathcal{I}$ we have

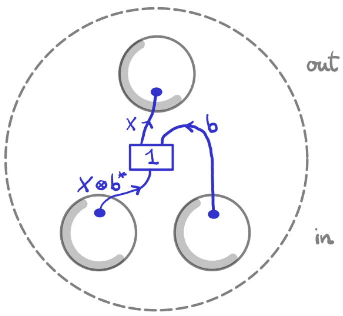
$\tau_{\mathcal{C}}\big(\emptyset \xrightarrow{M} S(a \otimes b^*) \xrightarrow{N} \emptyset\big)$

$= \dfrac{\operatorname{Dim}\mathcal{C}}{\mathcal{D}} \dfrac{\delta_{a,b}}{\dim a} \tau_{\mathcal{C}}\big(\emptyset \xrightarrow{M} S(a \otimes a^*) \xrightarrow{\quad}$  $\emptyset\big)\, \tau_{\mathcal{C}}\big(\emptyset \xrightarrow{\quad}$  $S(a \otimes a^*) \xrightarrow{N} \emptyset\big)$.

(How does the corollary follow from Lemma 5.111? Where does the factor of $\dim a$ come from?) (?)

■ Define $H_a : S(X) \to S(X \otimes a^*) \sqcup S(a)$ and $H'_b : S(X \otimes b^*) \sqcup S(b) \to S(X)$ as

$H_a =$ 

$H'_b =$ 

where in both cases the underlying manifold is $S^3 \backslash (B^3 \sqcup B^3 \sqcup B^3)$. Abbreviate

$$x = [S(X \otimes a^*) \xrightarrow{M_1} \Sigma_1] \ , \qquad y = [S(a) \xrightarrow{M_2} \Sigma_2] \ ,$$
$$u = [\Sigma_1 \xrightarrow{N_1} S(X \otimes b^*)] \ , \qquad v = [\Sigma_2 \xrightarrow{N_2} S(b)] \ .$$

Consider $\tilde{\phi}_{\Sigma_1,\Sigma_2}$ defined on a basis as

$$\begin{aligned} \tilde{\phi}_{\Sigma_1,\Sigma_2} : \bigoplus_{a \in \mathcal{I}_{\text{trans}}} V_{X \otimes a^*}(\Sigma_1) \otimes_{\Bbbk} V_a(\Sigma_2) \ &\to \ \mathbb{V}(X, \Sigma_1 \sqcup \Sigma_2) \\ x \otimes y \ &\mapsto \ [\![S(X) \xrightarrow{H_a} S(X \otimes a^*) \sqcup S(a) \xrightarrow{M_1 \sqcup M_2} \Sigma_1 \sqcup \Sigma_2]\!] \ . \end{aligned}$$

Dually, define $\tilde{\phi}'_{\Sigma_1,\Sigma_2}$ as

$$\begin{aligned} \tilde{\phi}'_{\Sigma_1,\Sigma_2} : \bigoplus_{b \in \mathcal{I}_{\text{trans}}} V'_{X \otimes b^*}(\Sigma_1) \otimes_{\Bbbk} V'_b(\Sigma_2) \ &\to \ \mathbb{V}'(X, \Sigma_1 \sqcup \Sigma_2) \\ u \otimes v \ &\mapsto \ \dim b \, [\![\Sigma_1 \sqcup \Sigma_2 \xrightarrow{N_1 \sqcup N_2} S(X \otimes b^*) \sqcup S(b) \xrightarrow{H'_b} S(X)]\!] \ . \end{aligned}$$

The factor of $\dim b$ will cancel a corresponding factor in the next lemma, which is the first key technical lemma which enables the construction of $\tau^2_{\Sigma_1,\Sigma_2}$:

Lemma 5.113. For u, v, x, y as above, we have

$$\langle\!\langle \tilde{\phi}'_{\Sigma_1,\Sigma_2}(u \otimes v), \tilde{\phi}_{\Sigma_1,\Sigma_2}(x \otimes y) \rangle\!\rangle_X = \delta_{a,b} \frac{\operatorname{Dim} \mathcal{C}}{\mathcal{D}} \langle u, x \rangle_{X \otimes a^*} \langle v, y \rangle_a \ .$$

Proof. The left hand side of the claim is by definition equal to

$$\begin{aligned} \text{lhs} = \dim(b) \, \tau_{\mathcal{C}} \Big(\operatorname{tr}_{S(X)} \Big(S(X) \xrightarrow{H_a} S(Xa^*) \sqcup S(a) \xrightarrow{M_1 \sqcup M_2} \Sigma_1 \sqcup \Sigma_2 \\ \xrightarrow{N_1 \sqcup N_2} S(Xb^*) \sqcup S(b) \xrightarrow{H'_b} S(X) \Big) \Big) \ . \end{aligned}$$

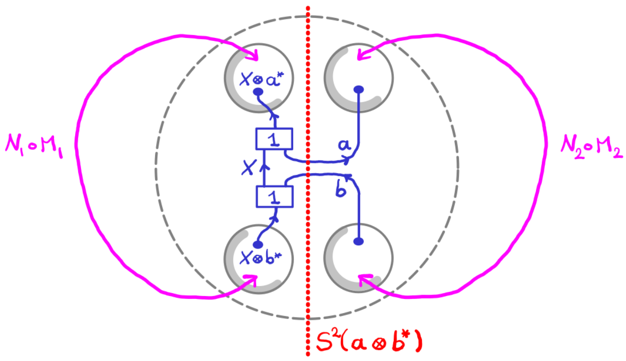
Define the bordism

$$Q = \Big[S(Xb^*) \sqcup S(b) \xrightarrow{H'_b} S(X) \xrightarrow{H_a} S(Xa^*) \sqcup S(a) \Big] \ .$$

Using cyclicity of the trace, we can rewrite

$$\begin{aligned} \text{lhs} = \dim(b) \, \tau_{\mathcal{C}} \Big(\operatorname{tr}_{S(Xb^*) \sqcup S(b)} \Big(S(Xb^*) \sqcup S(b) \xrightarrow{Q} S(Xa^*) \sqcup S(a) \\ \xrightarrow{(N_1 \circ M_1) \sqcup (N_2 \circ M_2)} S(Xb^*) \sqcup S(b) \Big) \Big) \ . \end{aligned}$$

Let R be the closed 3-manifold on which $\tau_{\mathcal{C}}$ is being evaluated above. R can schematically be written as follows:



Here, Q is shown in detail as S^3 with four 3-balls removed, and it is indicated how $N_1 \circ M_1 : S(Xa^*) \to S(Xb^*)$ and $N_2 \circ M_2 : S(a) \to S(b)$ are glued to the boundary 2-spheres. We can decompose R along the surface indicated by the dashed line into $R = \big[\emptyset \xrightarrow{R_1} S(a \otimes b^*) \xrightarrow{R_2} \emptyset\big]$ and apply Corollary 5.112 to obtain

$$\text{lhs} = \frac{\operatorname{Dim}\mathcal{C}}{\mathcal{D}}\, \delta_{a,b}\, \tau_{\mathcal{C}}\big(\operatorname{tr}_{S(Xb^*)}\big(S(Xa^*) \xrightarrow{N_1 \circ M_1} S(Xa^*)\big)\big)$$
$$\tau_{\mathcal{C}}\big(\operatorname{tr}_{S(a)}\big(S(a) \xrightarrow{N_2 \circ M_2} S(a)\big)\big) \ .$$

The two invariants $\tau_{\mathcal{C}}(\cdots)$ are equal to $\langle u, x\rangle_{X \otimes a^*}$ and $\langle v, y\rangle_a$, respectively. □

Lemma 5.114. $\tilde{\phi}_{\Sigma_1,\Sigma_2}$ induces a linear map

$$\phi_{\Sigma_1,\Sigma_2} : \bigoplus_{a \in \mathcal{I}_{\text{trans}}} \mathbb{V}(X \otimes a^*, \Sigma_1) \otimes_{\Bbbk} \mathbb{V}(a, \Sigma_2) \to \mathbb{V}(X, \Sigma_1 \sqcup \Sigma_2) \ .$$

Proof. By definition, $\mathbb{V}(Y, \Sigma) = V_Y(\Sigma)/N_Y(\Sigma)$, cf. page 178. Then (Why?) ?

$$\mathbb{V}(X \otimes a^*, \Sigma_1) \otimes_{\Bbbk} \mathbb{V}(a, \Sigma_2) = \frac{V_{X\otimes a^*}(\Sigma_1) \otimes_{\Bbbk} V_a(\Sigma_2)}{V_{X\otimes a^*}(\Sigma_1) \otimes_{\Bbbk} N_a(\Sigma_2) \oplus N_{X\otimes a^*}(\Sigma_1) \otimes_{\Bbbk} V_a(\Sigma_2)} \ .$$

For the existence of ϕ_{Σ_1,Σ_2} we thus need to check that $\tilde{\phi}_{\Sigma_1,\Sigma_2}$ vanishes on the two summands $V_{X\otimes a^*}(\Sigma_1) \otimes_{\Bbbk} N_a(\Sigma_2)$ and $N_{X\otimes a^*}(\Sigma_1) \otimes_{\Bbbk} V_a(\Sigma_2)$. But this is immediate from Lemma 5.113. (Why?) □ ?

Analogously one can define $\phi'_{\Sigma_1,\Sigma_2}$ in terms of $\tilde{\phi}'_{\Sigma_1,\Sigma_2}$, we skip the details.

■ We need a linear algebra statement:

Lemma 5.115. For $i = 1, 2$, let A_i, A'_i be k-vector spaces and $B_i : A'_i \times A_i \to k$ be non-degenerate bilinear forms. Then also

$$B : (A'_1 \otimes_k A'_2) \times (A_1 \otimes_k A_2) \to k \ , \quad B(a \otimes b, c \otimes d) = B_1(a, c)B_2(b, d) \ ,$$

is non-degenerate.

Proof. Non-degeneracy of B_i in the first argument is equivalent to $B_i^\# : A'_i \to (A_i)^*$, $a' \mapsto B_i(a', -)$ being injective. But then also $B_1^\# \otimes_k B_2^\# : A'_1 \otimes_k A'_2 \to A_1^* \otimes_k A_2^*$ is injective. Note that $B^\# : A'_1 \otimes_k A'_2 \to (A_1 \otimes A_2)^*$ satisfies

$$\begin{aligned} \big(B^\#(a \otimes b)\big)(c \otimes d) &= B(a \otimes b, c \otimes d) = B_1(a, c)B_2(b, d) \\ &= \big((B_1^\#(a)\big)(c)\big((B_2^\#(b)\big)(d) \ , \end{aligned}$$

which shows that $B^\# = B_1^\# \otimes_k B_2^\#$. Thus $B^\#$ is injective and hence B is non-degenerate in the first argument. Non-degeneracy in the second argument can be seen analogously. □

Note that for $x = [\![S(X \otimes a^*) \xrightarrow{M_1} \Sigma_1]\!]$, $u = [\![\Sigma_1 \xrightarrow{N_1} S(X \otimes b^*)]\!]$, etc., by construction we have

$$\langle\!\langle \phi'_{\Sigma_1,\Sigma_2}(u \otimes v), \phi_{\Sigma_1,\Sigma_2}(x \otimes y) \rangle\!\rangle_X = \delta_{a,b} \frac{\mathrm{Dim}\, \mathcal{C}}{\mathcal{D}} \langle\!\langle u, x \rangle\!\rangle_{X \otimes a^*} \langle\!\langle v, y \rangle\!\rangle_a \ .$$

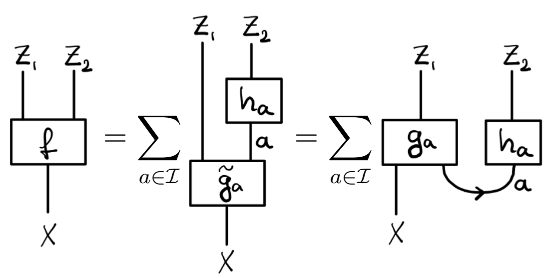
By the above lemma, as the two pairings on the right hand side are non-degenerate, so is the pairing on the left hand side. We get the following corollary:

Corollary 5.116. ϕ_{Σ_1,Σ_2} is injective.

The second key lemma for the construction of $\tau^2_{\Sigma_1,\Sigma_2}$ is:

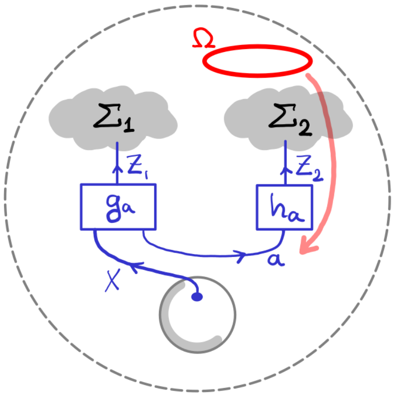
Lemma 5.117. ϕ_{Σ_1,Σ_2} is surjective.

Proof (sketch). By surjectivity of Φ_X (Proposition 5.94), every element in $\mathbb{V}(X, \Sigma_1 \sqcup \Sigma_2)$ can be written as $[\![B(X, \Sigma_1 \sqcup \Sigma_2, f)]\!]$ for $f \in \mathcal{C}(X, Z_1 \otimes Z_2)$, where Z_1, Z_2 are the contributions from Σ_1 and Σ_2 and can be read off from the proposition. We rewrite



for suitable morphisms $\tilde{g}_a$, g_a and h_a. Substituting this into $B(X, \Sigma_1 \sqcup \Sigma_2, f)$ gives:

$$[\![B(X, \Sigma_1 \sqcup \Sigma_2, f)]\!] = \frac{1}{\operatorname{Dim}\mathcal{C}} \sum_{a \in \mathcal{I}} [\![M_a]\!] \ , \quad M_a =$$



There the shapes labelled Σ_1 and Σ_2 represent the cut-out handle bodies for the connected components of each of Σ_1 and Σ_2, and Z_1 and Z_2 collect all the corresponding ribbons. The loop labelled by the surgery colour cancels against the prefactor $1/\operatorname{Dim}\mathcal{C}$. On the other hand, we can move the loop over the shape labelled Σ_2 and over the coupon h_a to form a small loop around the ribbon labelled a. By Lemma 5.49 (1), this restricts the sum over a from $\mathcal{I}$ to $\mathcal{I}_{\text{trans}}$. Comparing to the definition of ϕ_{Σ_1,Σ_2}, we can now read off that

$$[\![B(X, \Sigma_1 \sqcup \Sigma_2, f)]\!] = \phi_{\Sigma_1,\Sigma_2}\Big(\sum_{a \in \mathcal{I}_{\text{trans}}} \Phi_{Xa^*}(g_a) \otimes_{\Bbbk} \Phi_a(h_a) \Big) \ .$$

This shows that ϕ_{Σ_1,Σ_2} is surjective. □

- Combining the isomorphisms we found so far, we obtain

$$\begin{aligned}
\mathcal{S}\big(X, \tau_{\mathcal{C}}(\Sigma_1) \otimes_{\mathcal{S}} \tau_{\mathcal{C}}(\Sigma_2)\big) &\cong \mathcal{S}^{\boxtimes 2}\big(L(X), \tau_{\mathcal{C}}(\Sigma_1) \boxtimes \tau_{\mathcal{C}}(\Sigma_2)\big) \\
&\cong \bigoplus_{a \in \mathcal{I}_{\text{trans}}} \mathcal{S}^{\boxtimes 2}\big((X \otimes a^*) \boxtimes a, \tau_{\mathcal{C}}(\Sigma_1) \boxtimes \tau_{\mathcal{C}}(\Sigma_2)\big) \\
&\cong \bigoplus_{a \in \mathcal{I}_{\text{trans}}} \mathcal{S}(X \otimes a^*, \tau_{\mathcal{C}}(\Sigma_1)) \otimes_{\Bbbk} \mathcal{S}(a, \tau_{\mathcal{C}}(\Sigma_2)) \\
&\cong \bigoplus_{a \in \mathcal{I}_{\text{trans}}} \mathbb{V}(X \otimes a^*, \Sigma_1) \otimes_{\Bbbk} \mathbb{V}(a, \Sigma_2) \\
&\overset{(*)}{\cong} \mathbb{V}(X, \Sigma_1 \sqcup \Sigma_2) \\
&\cong \mathcal{S}\big(X, \tau_{\mathcal{C}}(\Sigma_1 \sqcup \Sigma_2)\big) \ ,
\end{aligned}$$

where $(*)$ is the isomorphism ϕ_{Σ_1,Σ_2} we just constructed. These isomorphism are all natural in X, and so by Remark 5.103 (3) define a unique isomorphism between the representing objects

$$\tau^2_{\Sigma_1,\Sigma_2} : \tau_{\mathcal{C}}(\Sigma_1) \otimes_{\mathcal{S}} \tau_{\mathcal{C}}(\Sigma_2) \xrightarrow{\sim} \tau_{\mathcal{C}}(\Sigma_1 \sqcup \Sigma_2) \ .$$

We have now gathered all the ingredients to formulate the main theorem of this section:

Theorem 5.118. Let $\mathcal{C}$ be an anomaly-free twist-nondegenerate ribbon fusion category over an algebraically closed field with choice of square root $\mathcal{D}$. Then

$$\tau_{\mathcal{C}} : \mathsf{Bord}_3(\mathcal{C}) \to Z_2(\mathcal{C})$$

is a topological field theory (that is, a symmetric monoidal functor).

We still need to prove that the isomorphism $\tau^2_{\Sigma_1,\Sigma_2}$ satisfies the coherence diagrams of a monoidal functor and that the resulting monoidal structure on $\tau_{\mathcal{C}}$ is symmetric, but we will skip these steps.

Exercise 5.119. TODO: Dimensional reduction. L gives L, modular category gives fusion ring

5.9 Mapping class group representations

Let $\Sigma, \Sigma' \in \mathsf{Bord}_3(\mathcal{C})$ and $\phi : \Sigma \to \Sigma'$ an isomorphism of $\mathcal{C}$-marked surfaces.

■ Given $\phi, \phi' : \Sigma \to \Sigma'$ we say ϕ and ϕ' are *isotopic* if ϕ can be deformed to ϕ' through PL-isomorphisms $\Sigma \to \Sigma'$. That is, there is a PL-map $\gamma : [0,1] \times \Sigma \to \Sigma'$ such that each γ_t is a PL-isomorphism and $\gamma_0 = \phi$, $\gamma_1 = \phi'$.

■ The *mapping cylinder* $M(\phi)$ of ϕ is the bordism $\Sigma \to \Sigma'$ given by $M(\phi) = \Sigma' \times [0,1]$ with parametrisations $\phi : \Sigma \to \Sigma' \times \{0\}$ and $\mathrm{id} : \Sigma' \to \Sigma' \times \{1\}$. Recall that the morphisms in $\mathsf{Bord}_3(\mathcal{C})$ are isomorphism classes of bordisms. We have:

- For two isomorphisms $\phi : \Sigma \to \Sigma'$ and $\psi : \Sigma' \to \Sigma''$ we have

$$[M(\psi)] \circ [M(\phi)] = [M(\psi \circ \phi)] : \Sigma \to \Sigma''$$

- $[M(\phi)] : \Sigma \to \Sigma'$ only depends on the isotopy class of ϕ, i.e. if $\phi, \phi' : \Sigma \to \Sigma'$ are isotopic, then $[M(\phi)] = [M(\phi')]$.

(Why do these hold? *Hint:* For the first point consider an isomorphism of bordisms which is $(x,t) \mapsto (\psi(x),t)$ on $M(\phi)$ and the identity on $M(\psi)$. For the second point, take an isotopy γ from ϕ to ϕ' and consider the map $\Sigma \times [0,1] \to \Sigma \times [0,1]$, $(x,t) \mapsto (\phi^{-1} \circ \gamma_t(x), t)$. For which boundary parametrisation is that an isomorphism of bordisms? Combine this with the first point.)

- For $\Sigma = \Sigma'$ we define the *mapping class group* of Σ to be

$$\mathrm{MCG}(\Sigma) = \mathrm{Isom}(\Sigma)/\mathrm{Isom}_0(\Sigma) \ ,$$

where $\mathrm{Isom}(\Sigma)$ is the set of PL-isomorphisms of the $\mathcal{C}$-marked surface Σ, and $\mathrm{Isom}(\Sigma)_0$ are those which are isotopic to the identity map (i.e. which can be continuously deformed in $\mathrm{Isom}(\Sigma)$ to $\mathrm{id} : \Sigma \to \Sigma$). (Check that $\mathrm{Isom}(\Sigma)_0$ is a normal subgroup of $\mathrm{Isom}(\Sigma)$, so that $\mathrm{MCG}(\Sigma)$ is indeed a group.)

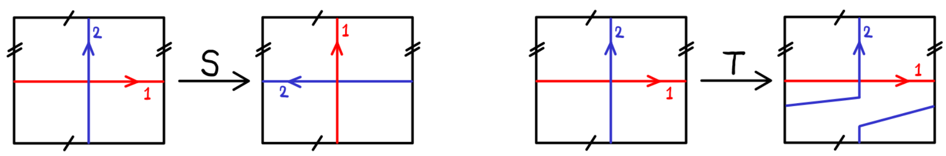
(For info: The mapping class groups are usually defined in terms of diffeomorphisms, not PL-isomorphisms. But in dimension 2 this does not make a difference. See [FM, Sec. 1.4.2] which compares diffeomorphism and homeomorphisms.)

Example 5.120. 1. Let Σ be $S^2 = \mathbb{R}^2 \cup \{\infty\}$ with marked points $1, \dots, n$ all labelled by $X \in \mathcal{C}$ and ∞ labelled by $Y \neq X$, the mapping class group $\mathrm{MCG}(\Sigma)$ is the framed braid group in n strands. See [FM, Sec. 9.1.3].

2. $\mathrm{MCG}(T^2) \cong SL(2,\mathbb{Z})$ is called the *modular group*. The matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

generate $SL(2,\mathbb{Z})$, and they correspond to the following isomorphisms of T^2:



From the discussion so far we get the following proposition:

Proposition 5.121. Let $\mathcal{C}$ be an anomaly-free twist-nondegenerate ribbon fusion category over an algebraically closed field. Then for each $\mathcal{C}$-marked surface Σ we obtain a representation of the mapping class group $\mathrm{MCG}(\Sigma)$ in $\mathcal{S} = Z_2(\mathcal{C})$:

$$\mathrm{MCG}(\Sigma) \to \mathrm{Aut}_{\mathcal{S}}(\tau_{\mathcal{C}}(\Sigma)) \quad , \quad \phi \mapsto \tau_{\mathcal{C}}(M(\phi)) \ .$$

Example 5.122. 1. $\mathcal{C}$ *is symmetric:* Consider the example of the framed braid group in n strands. The representation is by automorphisms acting on $X^{\otimes n} \otimes Y$, and it factors through the symmetric group of n elements. For a surface Σ without marked points, $\tau_{\mathcal{C}}(\Sigma) = \mathbb{1}$ and all $M(\phi)$ act as $1_{\mathbb{1}}$. (Why?) ?

2. $\mathcal{C}$ *is modular:* We obtain an action of the modular group $SL(2,\mathbb{Z})$ on the vector space $\tau_{\mathcal{C}}(T^2) = \mathcal{C}(\mathbb{1}, E)$ with $E = \bigoplus_{a \in \mathcal{I}} a^* \otimes a$. This is actually the reason why a modular fusion category is called "modular". Modular fusion categories are an important source of linear representations of surface mapping class groups.

Bibliographic notes for Section 5

Some textbook references on ribbon categories are [Tu, BK, TV, GP]. The Reshetikhin-Turaev functor $F_{\mathcal{C}} : \mathsf{Rib}_{\mathcal{C}} \to \mathcal{C}$ is due to [RT1]. The graphical calculus for ribbon categories is also developed in [Shu].

The standard source for facts about fusion categories is the book [EGNO]. The description of braiding and associator via F- and R-matrices appears, to my knowledge, first in [MSe].

The use of Kirby-calculus to define 3-manifold invariants goes back to [RT2] and the attendant 3d TFT has been constructed in [RT2, Tu]. The universal construction is due to [BHMV] and is also described for example in [Co] and [GP]. Reshetikhin-Turaev TFT has been described via the universal construction in [DGGPR] (even for non-semisimple input).

Using the universal construction to obtain a surgery TFT from a ribbon category which takes values in the symmetric centre of that category (Sections 5.7 and 5.8) is new. An analogous TFT has been obtained in [La], though not via the universal construction, and without embedded ribbon graphs.

For more on mapping class groups and their generalisation to modular functors see [BK].

6 Outlook

There is a very large range of topics not covered in these notes. Here are some examples:

- The three manifold invariants and TFT from Section 5.6 onwards were developed for fusion categories. But after being suitably adapted, the same basic construction also applies to non-semisimple tensor categories (at the point of writing, for the universal construction one needs to require modularity). This is nice as representation theory provides many interesting examples of such tensor categories. See [DGGPR, GP].

- In two dimensions we described a state-sum construction of a TFT, while in three dimensions we looked at a surgery construction. There is also a state-sum construction in three dimensions, called Turaev-Viro-Barrett-Westbury TFT, whose algebraic input is a spherical fusion category $\mathcal{F}$. The resulting TFT is actually equivalent to the surgery TFT for the modular fusion category $\mathcal{Z}(\mathcal{F})$, the so-called Drinfeld centre of $\mathcal{F}$. See [TV] for more on this.

- One can combine the state-sum construction developed in 2d in Section 4.5 with the surgery invariants and TFT from Section 5 to obtain 3d invariants and TFTs which do not only have 1d defects (the embedded ribbon graphs), but also 2d defect surfaces. These invariants can detect (in some cases) non-isotopically embedded surfaces. See [CRS2]. They also relate to interesting algebra, namely a categorification of modules over a ring.

- Starting in four dimensions the algebraic treatment of invariants becomes increasingly unwieldy. This is maybe not surprising given the jumps in effort we observed as we went up in dimension from one to two to three. Still, there is a well-developed theory of fusion-2-categories and a corresponding state-sum construction, called Douglas-Reutter TFTs. See [DR].

- Also starting in four dimensions, being continuous, PL, and smooth are no longer the same, and it is a question in current research to find TFTs which can distinguish between different smooth structures on the same underlying topological manifold. See [RW].

- Another approach to TFTs is via skein theory (also called string nets in two dimensions). Here one assigns vector spaces to manifolds by considering all embedded graphs modulo local relations. This has been done originally for the Kauffman bracket [BHMV]. For more general input categories in dimensions 3 and 4, see for example [Ki2, MW, CGHP, Wed], for stratified bordisms see [BJ]. String nets describe ground states of gapped Hamiltonians and in this way are related to topological phases of matter [LW].

- We did not touch the world of fully extended topological field theories and (∞, n)-categories. This provides the appropriate (and surprising) generalisation to any dimension of the observation in Section 3.1 that 1d TFTs are the same as fully dualisable objects in the target by evaluating on a point. See [BD, Lu1] and e.g. [Te, Fr, AKMT].

- One can consider n-dimensional TFTs with $(n-1)$-dimensional boundaries that are not topological. This goes back to the famous paper [Wi] relating 3d TFTs and the Jones polynomial. It has been used in [FRS] to understand and classify certain types of two-dimensional conformal field theories in terms of properties of the 3d TFT on whose boundary they are defined. See [vV+, pp. 36–57] for short lecture notes on this. More recently, this point of view has received a lot of attention as a description of symmetries of quantum field theory, see [CDR] and references therein for an overview.

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